MATH 323 Section 2

$\mathcal{TEST}$ 1 Solutions

February 1^{st}*, 2013*

1. (15pts) Use a truth table to show that $p \Rightarrow (q \Rightarrow r)$ is equivalent to $(q \wedge \sim r) \Rightarrow \sim p$

Solution: Here are two truth tables.

p	q	r	$p \Rightarrow$	$(q \Rightarrow r)$	p	q	r	$\sim p$	$\sim r$	$(q \wedge \sim r)$	$\Rightarrow \sim p$
T	T	T	T	T	T	T	T	F	F	F	T
T	T	F	F	F	T	T	F	F	T	T	F
T	F	T	T	T	T	F	T	F	F	F	T
T	F	F	T	T	T	F	F	F	T	F	T
F	T	T	T	T	F	T	T	T	F	F	T
F	T	F	T	F	F	T	F	T	T	T	T
F	F	T	T	T	F	F	T	T	F	F	T
F	F	F	T	T	F	F	F	T	T	F	T

We see that both are true unless p and q are true and r is false.

2. (15pts) Consider the following statement: If $a < b + \varepsilon$ for every $\varepsilon > 0$, then $a \leq b$.

a. (5pts) State the contrapositive of this statement.

Answer: If $a > b$ then there exists $\varepsilon > 0$ such that $a \geq b + \varepsilon$.

b. (10pts) Using the contrapositive, prove the statement is true. (Hint: consider the number $a - b$.)

Answer: Suppose $a > b$. Then $a - b > 0$ and so if we let $\varepsilon = a - b$ then $b + \varepsilon = a$.

3. (32pts) Consider the functions $f_1(x) = x^2$ and $f_2(x) = x^3$. For each of the following, show which of f_1 and f_2 satisfy the condition and which do not (it could be that both or neither satisfy it). Justify your answer. (Notice the different point values assigned to each part.)

a. (10pts) For all y there exists an x such that $f(x) \geq y$.

Answer: Both satisfy this. It means that given any y, we can find a x such that $f(x)$ is larger than it. Since both numbers get arbitrarily large, this can be done. For instance, given y, we can take $x = |y| + 1$ for each.

b. (10pts) There exists y such that for all x, $f(x) \geq y$.

Answer: This is true for f_1, since $f_1(x) \geq 0$ for all x. However, $f_2(x)$ gets arbitrarily small, so this is impossible.

c. (6pts) For all x there exists a y such that $f(x) \geq y$.

Answer: This is true for both, since given a value $f(x)$, since it is a real number there is some number smaller than it (like take $y = f(x) - 1$).

d. **(6pts)** There exists x such that for all y, $f(x) \geq y$.

Answer: This cannot be true for any functions since it means that there is an x such that $f(x)$ is larger than all numbers, which is impossible.

4. (21pts) Consider the statement "Every differentiable function is continuous."
a. If we think this statement is false, explain how one might go about trying to prove that it is false (simply give an idea of how to start and what would need to be accomplished).

Answer: We would need to find a differentiable function that is not continuous. This is a counterexample.
b. To prove this statement is true, explain how one would go about a direct proof of the statement.

Answer: We suppose f is a differentiable function. Then we need to prove that f is continuous.
c. Explain how one would go about proving the statement using its contrapositive.

Answer: The contrapositive is that every function that is not continuous is not differentiable, so we would suppose f is a function that is not continuous, and then we would need to show that f is not differentiable.

5. (20pts) Recall that an integer x is even if there exists an integer m such that $x = 2m$. Prove that for any integer y, y is even iff y^2 is even. (If you wish, you may use the fact that an integer z is not even iff it is odd, i.e., there exists an integer n such that $z = 2n + 1$.)

Answer: First we will show that if y is even, then y^2 is even. Suppose y is even. Then there exists an integer m such that $y = 2m$. Then

$$y^2 = (2m)^2 = 4m^2 = 2\left(2m^2\right).$$

Thus y^2 is even.

We will now show that if y is not even, then y^2 is not even. Suppose y is not even, then it is odd. Thus there exists an integer n such that $y = 2n + 1$. Then

$$y^2 = (2n+1)^2 = 4n^2 + 4n + 1 = 2\left(2n^2 + 2n\right) + 1.$$

Hence y^2 is odd, and thus not even.