## MATH 323 Section 2

# $\mathcal{TEST}$ 2

*February* $18^{st}$, *2013*

**1. (20pts)**
**a.** Use proof by contradiction to show that there is no $y \in \mathbb{R}$ such that

$$\frac{3y}{2+y} = 3.$$

Answer: Suppose $\frac{3y}{2+y} = 3$. Then

$$\begin{aligned} 3y &= 3(2+y) \\ 3y &= 6 + 3y \\ 0 &= 6. \end{aligned}$$

This is a contradiction.

**b.** Show that for each $x > 3$, there exists a $y$ such that

$$\frac{3y}{2+y} = x.$$

Answer: We notice that

$$\begin{aligned} 3y &= x(2+y) \\ (3-x)y &= 2x \\ y &= \frac{2x}{3-x} \end{aligned}$$

if $x > 3$. Thus there is a $y$ such that $\frac{3y}{2+y} = x$. (Note, this can be confirmed:

$$\frac{3\left(\frac{2x}{3-x}\right)}{2+\frac{2x}{3-x}} = \frac{\left(\frac{6x}{3-x}\right)}{\frac{6}{3-x}} = x,$$

but this may take more time than you had on the exam.)

**2. (25pts)** Suppose $A$, $B$, and $C$ are sets.
**a. (5pts)** Write what it means that $x \in A \cup (B \cap C)$
Answer: $x \in A$ or both $x \in B$ and $x \in C$.

**b. (5pts)** Write what it means that $x \in (A \cup B) \cap (A \cup C)$.
Answer: Both $x \in A$ or $x \in B$ and $x \in A$ or $x \in C$ are true. (or one could write ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$))

**c. (15pts)** Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
Answer: First we prove $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Let $x \in A \cup (B \cap C)$. Then $x \in A$ or both $x \in B$ and $x \in C$. In the first case, $x \in A$ so $x \in A \cup B$ and $x \in A \cup C$ as desired. In the second case, we also have $x \in A \cup B$ and $x \in A \cup C$, so $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
Now we prove $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Let $x \in (A \cup B) \cap (A \cup C)$. If $x \in A$, then we are done. If $x$ is not in $A$, then $x$ must be in both $B$ and $C$. Hence $x \in A \cup (B \cap C)$.

**3. (20pts)** Let $A$ and $B$ be the following sets:

$$A = \{x \in \mathbb{R} : (x+1)(3-x) \leq 0\}$$
$$B = \{y \in \mathbb{R} : y > 2\}.$$

Explain how one would prove the following facts (you do not need to actually prove them):

**a. (8pts)** $A \cup B$ and $[0,1]$ are disjoint.
Answer: The best answer is to suppose $x \in [0,1]$, show that $(x+1)(3-x) > 0$ and $x \leq 2$. One can prove this in other ways as well.

**b. (12pts)** $B \setminus A = \{z \in \mathbb{R} : 2 < z \leq 3\}$.

Note: The number 3 should not be in there, since $3 \in A$. The correct answer is $\{z \in \mathbb{R} : 2 < z < 3\}$. This was noticed by one student, who got bonus points.

Answer: We need to prove two things. First, suppose $x \in B$ and $x \notin A$ (that is, $x > 2$ and $(x+1)(3-x) > 0$.) and show that $2 < x < 3$. For the converse, suppose we have $z$ such that $2 < z \leq 3$. Then we need to show that $z > 2$ and $(x+1)(3-x) > 0$.

**4. (15pts)** Consider the relation defined on the set of integers given by $xRy$ iff there exists an integer $k$ such that $x + y = 3k$. (Note this is $x$ *plus* $y$, not $x$ *minus* $y$.) Show that this relation is symmetric but not reflexive or transitive.
Answer: The relation is symmetric because if $x + y = 3k$ then $y + x = x + y = 3k$. The relation is not reflexive; e.g., notice that $1 + 1 = 2 \neq 3k$ for any integer $k$. The relation is not transitive; consider $1R2$ and $2R4$, but 1 is not related to 4 since $1 + 4 = 5 \neq 3k$ for any integer $k$.

**5. (20pts)** Let $J$ be the index set defined by $J = \{x \in \mathbb{R} : 2 < x < 5\}$.
**a.** Show

$$\bigcap_{x \in J} [2, x] = \{2\}.$$

Answer: Since $2 \in [2, x]$ for all $x \in J$, we have that $\{2\} \subseteq \bigcap_{x \in J} [2, x]$. We now consider $y \in \mathbb{R}$ such that $y \neq 2$. If $y < 2$, then $y \notin [2, x]$ for any $x \in J$, in particular, it is not in $[2, 3]$

and hence is not in the intersection. If $y > 2$, then we can find an $x$ such that $2 < x < y$ (for instance, take $\frac{y-2}{2}$), and so for this particular $x$, $y \notin [2, x]$, and so $y$ is not in the intersection.

**b.** Show

$$\bigcup_{x \in J} [2, x] = [2, 5).$$

Answer: If $y \in \bigcup_{x \in J} [2, x]$, then $y \in [2, x]$ for some $x \in J$, and since $x < 5$, we have that $y \in [2, 5)$, and hence $\bigcup_{x \in J} [2, x] \subseteq [2, 5)$. Conversely, suppose $y \in [2, 5)$. If $y = 2$, $y \in [2, 3)$. If $y > 2$, since we also have $y < 5$, $y \in J$, and also $y \in [2, y)$. Thus $y \in \bigcup_{x \in J} [2, x]$.