Some solutions

Some things to be sure to know for Test 3 (note: this is not ALL you need for the test, but this is a good start)

1) Definitions of function, injective/injection, surjective/surjection, bijective/bijection, domain, codomain, range, image, preimage.

2) Definitions of equinumerous, finite, infinite, denumerable, countable, uncountable

3) Definitions of relation, reflexive, symmetric, transitive, equivalence relation, equivalence class.

4) If $f : A \to B$ and $C, C' \subseteq A$ and $D, D' \subseteq B$, then you should be able to prove basic things like:

$$\begin{aligned} C &\subseteq f^{-1}(f(C)) \\ f\left(f^{-1}(D)\right) &\subseteq D \\ f(C \cap C') &\subseteq f(C) \cap f(C') \\ f(C \cup C') &= f(C) \cup f(C') \\ f^{-1}(D \cap D') &= f^{-1}(D) \cap f^{-1}(D') \\ f^{-1}(D \cap D') &= f^{-1}(D) \cap f^{-1}(D') \end{aligned}$$

Recall that if there is an equal, there are two directions to prove. It is extremely important that you know how to start and end these proofs.

Answers: Some of these are on p. 66/72.

5) Basics of compositions of functions: Let $f : A \to B$ and $g : B \to C$ be functions. You should be able to prove:

a) If f and g are surjective, then $g \circ f$ is surjective.

Answer: Given $c \in C$, since g is surjective, there exists $b \in B$ sych that $g(b) = c$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Hence $g(f(a)) = c$.

b) If $g \circ f$ is surjective, then g is surjective, but f may not be.

Answer: If g is not surjective, then there exists $c \in C$ such that $g(b) \neq c$ for all $b \in B$. But then $g(f(a)) \neq c$ for all $a \in A$. Thus we have proven the contrapositive, and we find that if $g \circ f$ is surjective then g is surjective. It is possible that f is not. For instance, take $A = \{1, 2\}$, $B = \{1, 2, 3\}$, $C = \{1\}$, $f(x) = x$ and $g(y) = 1$ for all y. The function f is not surjective, but the composition is.

c) If f and g are injective, then $g \circ f$ is injective.
d) If $g \circ f$ is injective, then f is injective, but g may not be.

6) More things you should be able to prove:
a) Any finite set is not equinumerous to $\mathbb{N}$.
b) $\mathbb{N}$ is equinumerous to $\mathbb{N} \setminus \{1\}$.

Answer: The function $f : \mathbb{N} \to \mathbb{N} \setminus \{1\}$ given by $f(n) = n + 1$ is a bijection (inverse is $g(n) = n - 1$).

c) $\mathbb{N}$ is equinumerous to $\mathbb{Z}$.

Answer: We can consider the function $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} \frac{n}{2} \text{ if } n \text{ is even} \\ -\frac{n-1}{2} \text{ if } n \text{ is odd} \end{cases}$$

7) You should know the definition of arbitrary intersection and arbitrary unions. Also, you should be able to prove:
a) $\bigcap_{n \in \mathbb{N}} \left[0, \frac{1}{n}\right] = \{0\}$
b) $\bigcap_{n \in \mathbb{N}} (0, \frac{1}{n}] = \varnothing$
c) $\bigcup_{n \in \mathbb{N}} \left[0, 1 - \frac{1}{n}\right] = [0, 1)$
d) $\bigcup_{x \in (0,1)} \left[x, \frac{1}{x}\right] = (0, \infty)$