MATH 323 Section 2

$\mathcal{TEST}$ 3

March 22^{nd}, 2013

1. (10pts) Let $f : A \to B$ be a function. State what it means for f to be injective and what it means for f to be surjective.

Answer: The function f is injective if $f(x) = f(y)$ implies that $x = y$. The function is surjective if for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.

2. (22pts) Consider the relation on $\mathbb{N}$ given by aRb if there exists $k \in \mathbb{Z}$ such that $\frac{a}{b} = 2^k$.

a. (12pts) Show this is an equivalence relation.

Answer: Since $\frac{a}{a} = 1 = 2^0$, we have that the relation is reflexive. If $\frac{a}{b} = 2^k$, then $\frac{b}{a} = 2^{-k}$, so the relation is symmetric. Suppose $\frac{a}{b} = 2^k$ and $\frac{b}{c} = 2^{k'}$. Then

$$\frac{a}{c} = \frac{a}{b}\frac{b}{c} = 2^{k+k'}$$

and so the relation is transitive.

b. (10pts) Give an example of two different equivalence classes (that is, find $x, y \in \mathbb{N}$ such that $E_x \neq E_y$, where E_x and E_y are the equivalence classes of x and y, respectively).

Answer: We can see that $E_1 = \left\{2^k : k = 0, 1, 2, \ldots\right\}$ and $E_3 = \left\{3 \cdot 2^k : k = 0, 1, 2, \ldots\right\}$. These are not equivalent since there is no way to write 3 as a power of 2. Note that $E_1 = E_2$, so these would not be different classes.

3. (22pts) Let $f : A \to B$ and $g : B \to C$ be functions.

a. (12pts) Show that if $g \circ f$ is injective, then f is injective.

Answer: Suppose $g \circ f$ is injective. If $f(x) = f(y)$, then $g \circ f\ (x) = g \circ f\ (y)$. But since $g \circ f$ injective, it follows that $x = y$. Hence f is injective.

b. (10pts) Give an example of functions f and g such that $g \circ f$ is injective but g is not injective. Justify your answer.

Answer: There are many examples. One is $f : \{1, 2\} \to \{1, 2\}$ give by $f(x) = x$ and $g : \{1, 2, 3\} \to \{1, 2\}$ given by $g(1) = 1$, $g(2) = 2$, $g(3) = 2$. The composition is clearly $g \circ f$ $(x) = x$, which is injective.

4. (24pts) Let $f : A \to B$ be a function and let $S, T \subseteq A$ and $U, V \subseteq B$.

a. (12pts) Give a counterexample to the statement: if $f(S) \subseteq f(T)$, then $S \subseteq T$.

Answer: If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 2$ for all $x \in \mathbb{R}$, then if $S = \{1\}$ and $T = \{2\}$ then $f(S) = \{2\} = f(T)$ but S is not a subset of T. There are many other answers.

b. (12pts) Prove that if $U \subseteq V$, then $f^{-1}(U) \subseteq f^{-1}(V)$.

Answer: Suppose $U \subseteq V$. Let $x \in f^{-1}(U)$, so $f(x) \in U$. It follows from $U \subseteq V$ that $f(x) \in V$, so $x \in f^{-1}(V)$. Thus $f^{-1}(U) \subseteq f^{-1}(V)$.

5. (22pts) Consider the set of numbers $S = \left\{\frac{a}{2} + \frac{b}{3} : a, b \in \mathbb{Z}\right\}$.

a. (7pts) Show that $\mathbb{Z} \subseteq S$.

Answer: The easiest way to do this is to notice that for any $n \in \mathbb{Z}$ we have that $n = \frac{2n}{2} + \frac{0}{3}$, and thus $n \in S$. There are many other ways to prove this.

b. (15pts) Show that the set S is countable.

Answer: Again, there are several ways to prove this. The nicest I saw was to notice that $S \subseteq \mathbb{Q}$, and since $\mathbb{Q}$ is countable, S must be countable. Another way to do this is to recall that $\mathbb{Z} \times \mathbb{Z}$ is countable (since $\mathbb{Z}$ is countable and the product of two sets is countable) so there is a surjective function $f : \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$. Furthermore, there is a function $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{S}$ given by $g(a, b) = \frac{a}{2} + \frac{b}{3}$. Since both f and g are surjective, $g \circ f : \mathbb{N} \to S$ is surjective, and thus by our theorem, S is countable.

Extra Credit (10pts): you may do ONLY ONE of the following problems:

a. Find a bijection g between the set of even integers $E = \{2n : n \in \mathbb{Z}\}$ and the set of positive powers of 3, $P_3 = \{3^m : m \in \mathbb{N}\}$. Show it is a bijection.

Answer: We can define the function $g : E \to P_3$ by

$$g(x) = \begin{cases} 3^x \text{ if } x > 0 \\ 3^{1-x} \text{ if } x \leq 0 \end{cases}$$

We see that it is one-to-one since the positive evens go to different even powers of 3, and the negative evens go to odd powers of 3. All odds are represented and all evens are represented, so this map is a bijection.

b. Show $\bigcup_{x \in (0,1)} \left[x, \frac{1}{x}\right] = (0, \infty)$

Answer: Let $y \in (0, \infty)$. We do separately the cases $y < 1$, $y > 1$, and $y = 1$. If $y < 1$, then $\frac{1}{y} > 1$ and $y \in \left[y, \frac{1}{y}\right]$. If $y > 1$ then $\frac{1}{y} < 1$ and so $y \in \left[\frac{1}{y}, y\right]$. If $y = 1$, then we see that $y \in \left[\frac{1}{2}, 2\right]$. Thus we have that $y \in \bigcup_{x \in (0,1)} \left[x, \frac{1}{x}\right]$.

Conversely, suppose $y \leq 0$. Since for any $x \in (0, 1)$, $y \leq 0 < x$, we have that $y \notin \bigcup_{x \in (0,1)} \left[x, \frac{1}{x}\right]$.