MATH 323 Section 2

$\mathcal{TEST}$ 4

April 15^{th}, *2013*

Your Name: ____

Directions:

a. You may NOT use your book or your notes or a calculator.
b. Please ask for extra scrap paper if needed.
c. Good Luck!

1. (10pts) For the following sets, determine whether they are countable or uncountable. Justify your answer (but you do not need detailed proofs).

a. The set $[-1,1]\cup\mathbb{N}$.

Answer: This set is uncountable since $[-1,1]$ is uncountable (there is a bijection with $\mathbb{R}$, as described in the homework), and the union of a countable set and an uncountable set must be uncountable.

b. The set of solutions to quadratic equations with integer coefficients, that is,

$$\left\{x\in\mathbb{R} : \text{there exist } a,b,c\in\mathbb{Z} \text{ such that } ax^2+bx+c=0\right\}.$$

Answer: This is countable, since it is a subset of the set of algebraic numbers. Similarly, we could see that there is a surjection $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\to E$, where E is the set of possible equations. Since the first is countable, we must have that E is countable. Since there are at most 2 solutions per equations, we can order this so that the set in question is the countable union of countable (finite) sets.

2. (30pts) Let $f:A\to B$ be a function and $C,D\subseteq B$.

a. (15pts) Show that $f^{-1}(C\cap D)=f^{-1}(C)\cap f^{-1}(D)$

Answer: Let $x\in f^{-1}(C\cap D)$. So $f(x)\in C\cap D$, i.e., $f(x)\in C$ and $f(x)\in D$. But this implies that $x\in f^{-1}(C)$ and $x\in f^{-1}(D)$, thus $x\in f^{-1}(C)\cap f^{-1}(D)$.

Conversely, suppose $x\in f^{-1}(C)\cap f^{-1}(D)$, so $x\in f^{-1}(C)$ and $x\in f^{-1}(D)$. Thus $f(x)\in C$ and $f(x)\in D$, hence $f(x)\in C\cap D$ and $x\in f^{-1}(C\cap D)$.

b. (15pts) Show that f is surjective and $f^{-1}(C)\subseteq f^{-1}(D)$, then $C\subseteq D$.

Answer: Suppose f is surjective and $f^{-1}(C) \subseteq f^{-1}(D)$. Consider $x \in C$. Since f is surjective, there exists $y \in A$ such that $f(y) = x$, and since $x \in C$, we have $y \in f^{-1}(C)$. Thus $y \in f^{-1}(D)$ and so $x = f(y) \in D$.

3. (20pts) Prove that

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

for all $n \in \mathbb{N} \setminus \{1\}$.

Answer: We induct on n. For the basis, we note that $\left(1 - \frac{1}{2^2}\right) = \frac{3}{4} = \frac{2+1}{2(2)}$. Now suppose

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}$$

for some $k \geq 2$. We consider

$$\begin{aligned}
&\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{k^2}\right)\left(1 - \frac{1}{(k+1)^2}\right) \\
&= \frac{k+1}{2k}\left(1 - \frac{1}{(k+1)^2}\right) \\
&= \frac{k+1}{2k}\frac{k^2+2k}{(k+1)^2} \\
&= \frac{k+2}{2(k+1)}.
\end{aligned}$$

This completes the proof (using the principle of mathematical induction).

4. (10pts) Suppose we wanted to prove the fact that $2^n < n!$ if $n \geq 4$.
a. State and prove the basis for the induction.

Answer: The basis is $2^4 < 4!$, which is true since $2^4 = 16 < 24 = 4!$

b. Give the inductive hypothesis for a proof by induction. (Do not finish the proof.)

Answer: The inductive hypothesis is that $2^k < k!$ for some $k \geq 4$.

5. (30pts) For this problem, always refer to the field axioms and theorem by labels on the included handout.

a. (10pts) For the following proof, justify each line by labeling with the appropriate axioms/theorem part(s).

Theorem: $0 < 1$.

Proof: Suppose by contradiction that $1 < 0$. Then $-1 > 0$. [**Theorem e**]

It follows that $1 < 0$ implies that $(-1) < 0$. [**O4 and Theorem b**]

It then follows that $1 > 0$, a contradiction (since $1 < 0$). [**Theorem e**]

Prove ONLY ONE of the following, justifying your answers using only the field axioms and theorem:

b1. (20pts) For all $x \in S$, if $x \neq 0$ then $x^2 > 0$.

Answer: Suppose $x > 0$, then we can use O4 to see that $x^2 > 0 \cdot x = 0$ by Theorem b. Now suppose $x < 0$. Then by Theorem f we have that $x^2 > 0 \cdot x = 0$ (again by Theorem b). By trichotomy (O1), this completes the proof.

b2. (20pts) If $x > 0$ then $\frac{1}{x} > 0$. If $x < 0$ then $\frac{1}{x} < 0$.

Answer: Suppose $x > 0$ and $\frac{1}{x} < 0$. Then using O4 we would have $1 = \frac{1}{x}x < x \cdot 0 = 0$, contradicting the part a. Similarly, if $x > 0$ and $\frac{1}{x} > 0$ we have the same. (Note: if $x \neq 0$, we must have $\frac{1}{x} \neq 0$ since $x\frac{1}{x} = 1 \neq 0$, considering Theorem b).