Chapter Check for Chapter 2

October 2, 2015

1. Let $P_k(\mathbb{R})$ denote the set of polynomials in one variable of degree at most k with coefficients in $\mathbb{R}$. Let $Y = \{y_0, \ldots, y_k\}$ and recall that $\mathcal{F}(Y, \mathbb{R})$ is the vector space of real-valued functions from Y to $\mathbb{R}$.

a. Explain why $P_k(\mathbb{R})$ and $\mathcal{F}(Y, \mathbb{R})$ are isomorphic without finding an isomorphism (i.e., use a theorem).

b. Find an explicit isomorphism and show it is an isomorphism.

2. Consider the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ with ordered bases $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$ and $\gamma = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \\ 2x \end{pmatrix}.$$

a. Find $[T]_\beta^\gamma$.
b. Describe $N(T)$ and $R(T)$.
c. If E_2 and E_3 denote the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, find the change of coordinates matrices P taking E_2 to β and Q taking γ to E_3. Compute $Q[T]_\beta^\gamma P$ and compare to $[T]_{E_2}^{E_3}$.

3. (Comprehensive/graduate option only) The space F^n acts on F^n by left multiplication (as in matrix multiplication), in the following way: for each $v \in F^n$, define $f_v : F^n \to F$ by

$$f_v(x) = v^T x,$$

where v^T denotes the transpose of v (so v^T is a row vector). Show that the map $F^n \to (F^n)^*$ given by $v \to f_v$ is an isomorphism.