EXAM 1

October 14th, 2015

Your Name: ________________________________

Directions:

a. You may NOT use your book or your notes.
b. Please ask for extra scrap paper if needed.
c. Show all work. Unless otherwise noted, a solution without work is worth nothing.
d. Good Luck!

Score:

1. _________
2. _________
3. _________
4. _________
5. _________

Total _________ /100
1. (15pts) Let $V$ be a vector space (not necessarily finite-dimensional) and let $T : V \rightarrow V$ be a linear transformation. For each of the following, determine whether or not the condition implies that $T$ is (i) one-to-one, (ii) onto, and/or (iii) an isomorphism (note: an isomorphism from $V$ to itself is called an automorphism). Give short justifications.

   a. $R(T) = V$.

   b. nullity $(T) = 0$ and $V$ is finite dimensional.

   c. $V$ is finite dimensional and nullity $(T) + \text{rank} (T) = \dim V$. 
2. (20pts) Let $V$ be a vector space, let $T : V \to V$ be a linear transformation, and let $T_0 : V \to V$ denote the zero transformation (that is, $T_0(v) = \vec{0}$ for all $v \in V$). Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.
3. (20pts) Consider the linear transformation \( L_A : \mathbb{R}^4 \to \mathbb{R}^5 \) determined by the matrix

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Let \( E_n \) denote the standard basis for \( \mathbb{R}^n \) and \( \beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \) be an ordered basis of \( \mathbb{R}^4 \).

a. (10pts) Compute \([L_A]_{E^5}^{E_5}\).

b. (10pts) Compute the change of basis matrix \( Q \) taking \( \beta \) to \( E_4 \) and double check that \([L_A]_{E^5}^{E_6} = AQ\).
4. (25pts)
   a. (15pts) Let $V_1$ and $V_2$ be subspaces of $V$, a finite dimensional vector space. Show that $\dim(V_1 + V_2) \leq \dim V_1 + \dim V_2$

   b. (10pts) Let $T, U : V \rightarrow W$ be linear transformations. Prove that $R(T + U) \subseteq R(T) + R(U)$
5. (20pts) An $n \times n$ matrix is *unit upper triangular* if it has ones along the diagonal and zeroes below the diagonal, so it has the form

\[
\begin{pmatrix}
1 & a_{12} & a_{13} & \cdots & a_{1n} \\
0 & 1 & a_{23} & \cdots & a_{2n} \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & 1 & a_{(n-1)n} \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

a. (5pts) Explain why the set of all unit upper triangular matrices does not form a subspace of the vector space of $n \times n$ matrices.

b. (15pts) Prove that any unit upper triangular matrix is invertible.