MATH 413/513

EXAM 2

November 18th, 2015

Your Name: _______________________________________

Directions:

a. You may NOT use your book or your notes.
b. Please ask for extra scrap paper if needed.
c. Show all work. Unless otherwise noted, a solution without work is worth nothing.
d. The total possible points are 105, but your score will be counted out of 100.
e. Good Luck!

Score:

1. __________
2. __________
3. __________
4. __________
5. __________
Total __________ /100
1. (20pts) Suppose $A$ is a $10 \times 10$ matrix with entries in $\mathbb{C}$ and the rank of $A$ is 7. For the following, determine whether the statement is necessarily true, necessarily false, or there is not enough information to determine if it is true or false. Give short justifications.

a. $\det A = 0$

b. The reduced row echelon form of $A$ contains at least one row of zeroes.

c. There exists an invertible matrix $Q$ such that $Q^{-1}AQ$ is diagonal.

d. There exist invertible matrices $Q$ and $P$ such that $QAP$ is diagonal.

e. $A$ is a product of elementary matrices.
2. (20pts) Consider the matrix \( A = \begin{pmatrix} 1 & -3 & -7 & 5 & -4 \\ 2 & 2 & 2 & 6 & -8 \\ -1 & 2 & 5 & -2 & -1 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} \). Its reduced row echelon form is 
\[ \begin{pmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]. Answer the following questions about \( A \):

a. What is the rank of \( A \)?

\[ \text{a. The rank of } A = 3 \text{ since there are three non-zero rows.} \]

b. Give a basis for the column space of \( A \).

\[ \text{b. A basis for the column space of } A \text{ is given by the non-zero columns:} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 2 \\ 1 \end{pmatrix}. \]

c. Give a basis for the nullspace of \( A \).

\[ \text{c. A basis for the nullspace of } A \text{ is given by the non-zero vector:} \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}. \]

d. Suppose for some vector \( b \in \mathbb{R}^4 \), \( x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \) is a solution to \( Ax = b \). Find all solutions of \( Ax = b \) in \( \mathbb{R}^5 \).

\[ \text{d. To find all solutions of } Ax = b \text{, we express } x \text{ as a general solution:} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix}, \text{ where } t_1, t_2, t_3, t_4, t_5 \text{ are free variables.} \]
3. (20pts)

a. (10pts) Prove that an upper triangular $n \times n$ matrix is invertible if and only if all its diagonal entries are nonzero.

b. (10pts) Suppose a matrix $A$ is diagonalizable, so there is an invertible matrix $Q$ such that $Q^{-1}AQ$ is diagonal. Show that for each standard basis element $e_i$, the vector $Qe_i$ is an eigenvector for $A$. (Hint: each $e_i$ is an eigenvector for the diagonal matrix.)
4. (25pts) Let $T$ be an invertible linear operator on a finite-dimensional vector space $V$.

a. (15pts) Prove that the scalar $\lambda$ is an eigenvalue of $T$ if and only if $\lambda^{-1}$ is an eigenvalue of $T^{-1}$. Hint: Notice that $T - \lambda I_V = -\lambda T' (T^{-1} - \lambda^{-1} I_V)$.

b. (10pts) Prove that the eigenspace of $T$ corresponding to $\lambda$ is the same as the eigenspace of $T^{-1}$ corresponding to $\lambda^{-1}$ (Note: you may use the result of part a even if you cannot prove it.)
5. (20pts) Recall that an inner product space is a vector space $V$ over $\mathbb{R}$ together with a product $\langle \cdot, \cdot \rangle$ such that for all $x, y, z \in V$ and $c \in \mathbb{R}$,

- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$,
- $\langle cx, z \rangle = c \langle x, z \rangle$,
- $\langle x, z \rangle = \langle z, x \rangle$,
- $\langle x, x \rangle > 0$ if $x \neq \mathbf{0}$.

Consider the vector space $\mathbb{R}^{2 \times 2}$ of two-by-two matrices with real entries and define a product $\langle A, B \rangle = \text{tr} (B^T A)$, where $\text{tr}$ denotes the trace (recall that $\text{tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$). Show that this is an inner product space. You may use the following facts without proof:

1. $\mathbb{R}^{2 \times 2}$ is a vector space over $\mathbb{R}$.
2. The trace is linear: $\text{tr} (cA + B) = c \text{tr} (A) + \text{tr} (B)$ for all $c \in \mathbb{R}$ and $A, B \in \mathbb{R}^{2 \times 2}$.
3. The trace is invariant under transpose: $\text{tr} (A^T) = \text{tr} (A)$ for all $A \in \mathbb{R}^{2 \times 2}$.
4. $(AB)^T = B^T A^T$ for all $A, B \in \mathbb{R}^{2 \times 2}$. 