Math 443/543 Graph Theory Notes 3: Shortest Paths

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1 Shortest path problems and Dijkstra's algorithm

This section is from BM 1.8 (we will use ϕ instead of w). We consider the shortest path problem: Given a railway network connecting various towns, determine the shortest route between a given pair of towns.

Definition 1 *A* network *(or* weighted graph*) is a graph G together with a map $\phi : E \to \mathbb{R}$. The function ϕ may represent the length of an edge, or conductivity, or cross-sectional area or many other things.*

Given a network (G, ϕ), we can define the weight of a subgraph $H \subset G$ to be

$$\phi(H) = \sum_{e \in E(H)} \phi(e).$$

The problem is then: given two vertices $u_0, v_0 \in V(G)$, find a u_0v_0-path of smallest weight (where we consider the path as a subgraph).

NOTE: We will assume that $\phi(e) > 0$ for the remainder of the section, as this simplifies exposition.

Often, we will refer to the the weight of an edge as a length and the value of the smallest weight as the distance. We will present the algorithm of Dijkstra and Whiting-Hillier (found independently). In the sequel, we will assume that ϕ is defined on all pairs of vertices and $\phi(uv) = \infty$ if $uv \notin E(G)$.

Definition 2 *The* distance *between two vertices $u, v \in V(G)$ is equal to*

$$d(u, v) = d_G(u, v) = \min\{\phi(P) : P \text{ is a path from } u \text{ to } v\}.$$

A path P which attains the minimum is called a shortest path*.*

We then have the following algorithm, known as Dijkstra's algorithm:

1. Let $\ell(u_0) = 0$ and let $\ell(v) = \infty$ for all $v \neq u_0$. Let $S_0 = \{u_0\}$ and let $i = 0$.

2. For each $v \in S_i^c$, replace $\ell(v)$ with
$$\min_{u \in S_i} \{\ell(v), \ell(u) + \phi(uv)\}.$$

3. Compute M to be
$$M = \min_{v \in S_i^c} \{\ell(v)\}$$
and let u_{i+1} be the vertex which attains M.

4. Let $S_{i+1} = S_i \cup \{u_{i+1}\}$.

5. If $i = p - 1$, stop. If $i < p - 1$, then replace i with $i + 1$ and goto step 2.

Lemma 3 *If $v_0, v_1, \ldots, v_k$ is a shortest path, then $v_0, v_1, \ldots, v_j$ is a shortest path for any $j \leq k$.*

Proof. If there were a shorter path from v_0 to v_j, then we could replace the current path with a shorter beginning and get a shorter path to v_k. ■

Let's prove that at the termination of the algorithm, $\ell(u) = d(u, u_0)$. We will induct on i. Clearly, this is true for $i = 0$. We will make the following inductive hypothesis:

- For every $u \in S_i$, $\ell(u) = d(u, u_0)$.

We have the base case, so we need only prove the inductive step. Suppose it is true for S_i. We must show that
$$d(u_0, u_{i+1}) = \ell(u_{i+1}).$$
Let $P = v_0, v_1, v_2, \cdots, v_k$, where $v_0 = u_0$ and $v_k = u_{i+1}$, be a $u_0 u_{i+1}$-path such that
$$d(u_0, u_{i+1}) = \phi(P).$$
If $v_{k-1} \in S_i$, then the path $P' = v_0, v_1, v_2, \cdots, v_{k-1}$ is a shortest path and by the inductive hypothesis $\phi(P') = \ell(v_{k-1})$. Thus
$$d(u_0, u_{i+1}) = \phi(P) = \ell(v_{k-1}) + \phi(v_{k-1} u_{i+1}) \geq \ell(u_{i+1})$$
but since $d(u_0, u_{i+1})$ is the minimum length path and $\ell(u_{i+1})$ is the length of some path, then we must have equality. Thus the inductive step is proven if $v_{k-1} \in S_i$.

We now show $v_{k-1} \in S_i$. Take the smallest j such that $v_j \notin S_i$. Then since $P_j = v_0, v_1, \ldots, v_j$ is a shortest path, we have, since $v_{j-1} \in S_i$, that
$$\ell(v_j) \leq \ell(v_{j-1}) + \phi(v_{j-1} v_j) = \phi(P_j) \leq \phi(P) \leq \ell(u_{i+1}).$$
since $\ell(u_{i+1}) = \min\{\ell(u) : u \in S_i^c\}$, that means that all of the inequalities are equalities and $j = k$ (since $P_j = P$) and $v_{k-1} \in S_i$. By the previous argument, we are done.

See BM-1.8 for a discussion of the complexity of this algorithm. It turns out to be a good algorithm.