Math 443/543 Graph Theory Notes 5: Planar graphs and coloring

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1 Planar graphs

The Three Houses and Three Utilities Problem: Given three houses and three utilities, can we connect each house to all three utilities so that the utility lines do not cross.

We can represent this problem with a graph, connecting each house to each utility. We notice that this graph is bipartite:

Definition 1 *A* bipartite graph *is one in which the vertices can be partitioned into two sets X and Y such that every edge has one end in X and the other in Y. The partition (X, Y) is called a* bipartition*. A* complete bipartite graph *is one such that every vertex of X is joined with ever vertex of Y. The complete bipartite graph such that the order of X is m and the order of Y is n is denoted $K_{m,n}$ or $K(m,n)$.*

It is easy to see that the relevant graph in the problem above is $K_{3,3}$. Now, we wish to embed this graph in the plane such that no two edges cross except at a vertex.

Definition 2 *A* planar graph *is a graph which can be drawn in the plane such that no two edges cross except at a vertex. A planar graph drawn in the plane in a way such that no two edges cross except at a vertex is called a* plane graph*.*

Note that it says "can be drawn" not "is drawn." The problem is that even if

Thus the problem is whether or not $K_{3,3}$ is a planar graph.

A planar graph divides the plane into regions. That is, if we remove the vertices and edges from the plane, there are a number of disconnected pieces, each of which we call a *region*. The *boundary* of a given region is all of the edges and vertices incident on the region. Notice that there is always one *exterior region* which contains all of the unbounded parts of the plane.

Look at examples. Notice that $p - q + r = 2$. A theorem of Euler says that this is always true.

Theorem 3 (Euler's Theorem) *Let G be a connected plane graph with p vertices, q edges, and r regions. Then*

$$p - q + r = 2.$$

We remark that the number 2 has to do with the plane, and will become important when we look at topology.

Proof. We induct on q. If $q = 0$, then we must have $p = 1$ and $r = 1$. Thus $p - q + r = 1 - 0 + 1 = 2$. Now suppose it is true for all graphs with k or fewer edges. Consider a connected graph G with $k + 1$ edges. If G is a tree, then we know that $p = q + 1 = k + 2$. Furthermore, since there are no cycles in a tree, $r = 1$. Thus

$$p - q + r = (k + 2) - (k + 1) + 1 = 2.$$

If G is not a tree, then it contains a cycle. Let $e \in G$ be an edge on a cycle. If we look at $G' = G - e$, it has k edges, and hence $p' = p$, $q' = k = q - 1$, and by the inductive hypothesis,

$$p' - q' + r' = 2.$$

We see that $r' = r - 1$ since removing the edge e joins the regions on either side of the e. Thus

$$p - q + r = p' - (q' + 1) + (r' + 1) = 2.$$

■

Notice the following corollary:

Corollary 4 *Any representation of a planar graph as a plane graph has the same number of regions.*

We can now solve the Problem of Three Houses and Three Utilities.

Theorem 5 *The graph $K_{3,3}$ is not planar.*

Proof. Suppose we can represent $K_{3,3}$ as a plane graph. We know that $p = 6$ and $q = 9$, so we need to understand something about r. Since $K_{3,3}$ is bipartite, we see that all regions have boundaries with at least 4 edges. Suppose the graph divides the plane into regions $R_1, R_2, \ldots, R_r$. Let $B(R)$ be the number of edges in the boundary of region R. Consider the number

$$N = \sum_{i=1}^{r} B(R_i).$$

Since all regions have boundaries with at least 4 edges, we have

$$N \geq 4r.$$

However, since each edge is counted at most twice, we have that

$$N \leq 2q = 18.$$

Thus

$$4r \leq 18$$

or

$$r \leq 4.5.$$

However, that would mean that

$$p - q + r = 6 - 9 + r \leq 1.5,$$

which is impossible if the graph is a plane graph. ■

One might ask about other non-planar graphs. Another important one is K_5. Here is a theorem which allows us to show this.

Theorem 6 *Let G be a connected, planar graph with p vertices and q edges, with $p \geq 3$. Then*

$$q \leq 3p - 6.$$

Proof. The proof is quite similar to that of the previous theorem. For $p = 3$, we certainly have $q \leq 3$. For $p \geq 4$, we consider a representation of G as a plane graph, which gives us r regions, which we denote as $R_1, R_2, \ldots, R_r$. We compute again

$$N = \sum_{i=1}^{r} B\left(R_i\right).$$

This time, we note that $B\left(R\right) \geq 3$ for each region, and a similar argument give us that

$$3r \leq N \leq 2q.$$

Using Euler Theorem, we have that

$$\begin{aligned} 2 &= p - q + r \\ &\leq p - q + \frac{2}{3}q = p - \frac{1}{3}q \end{aligned}$$

or

$$q \leq 3p - 6.$$

■

Corollary 7 *The complete graph K_5 is not planar.*

Proof. We see that K_5 has $p = 5$ and $q = \binom{5}{2} = 10$, and so

$$q = 10 > 15 - 6 = 3p - 6.$$

■

2 Kuratowski's theorem

This exposition is mainly from BM-9.5. We saw that K_5 and $K_{3,3}$ are not planar. It easily follows that any graph containing one of these as a subgraph (technically, any graph with a subgraph isomorphic to one of these two graphs) is not planar either. In fact, we can do slightly better by seeing that any graph with a subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$ is not planar.

Definition 8 *A* subdivision *of a graph G is a new graph obtained by adding vertices inside the edges. (The new vertices are all of degree 2.)*

It is then clear that :

Proposition 9 *If a subdivision of G is planar, then G is planar.*

Proof. Given a representation of the subdivision of G as a plane graph, simply remove the vertices which form the subdivision and we arrive at a plane graph representation of G. ■

Stated another way, if G is not planar, then any subdivision is not planar. Thus any subdivision of K_5 and $K_{3,3}$ is nonplanar.

Proposition 10 *If G is planar, then every subgraph is planar.*

Proof. Represent G as a plane graph, then the subgraphs are also plane graphs. ■

Thus if a subgraph of G is a subdivision of K_5 or $K_{3,3}$, then it is not planar, and thus G is not planar.

Kuratowski's theorem is the converse, i.e., that if a graph is not planar, then it contains a subdivision of K_5 or $K_{3,3}$ as a subgraph (technically, contains a subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$. Proof is in BM.

3 Topology comments

Plane graphs are also graphs on spheres. Using stereographic projection, one can convert any plane graph to a graph on the sphere with no vertex at the north pole. Stereographic projection is a map from the sphere except the north pole to the plane which is a homeomorphism (i.e., a bijection which is continuous with continuous inverse). The map is defined as follows. Consider the plane in $\mathbb{R}^3$ and the xy-plane, $\mathbb{R}^2$. For any point P on the sphere, draw a line in $\mathbb{R}^3$ from the north pole to that point. It will intersect the plane $\mathbb{R}^2$ at a point P'. Stereographic projection is the map $P \to P'$. With some basic geometry, one can explicitly write down the map, which is

$$\phi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

The inverse map can also be written explicitly, as

$$\phi^{-1}(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right).$$

We have the following:

Proposition 11 *Let $v \in V(G)$ for a planar graph G. There is a plane graph representing G such that v is in the boundary of the exterior region.*

Proof. Take a plane graph representing G. Map it by inverse stereographic projection to the sphere. Some region contains v in its boundary. Rotate the sphere so that a point in that region is at the north pole, then use stereographic projection to project it back to the plane. ■

Recall that our proof of nonembeddability of $K_{3,3}$ and K_5 used Euler's theorem, which is essentially a theorem about topology. We can replace the use of Euler's theorem with a different topological theorem, the Jordan curve theorem.

Theorem 12 (Jordan curve theorem) *A Jordan curve is a curve in the plane with no self-intersection except that it begins and ends at the same point. A Jordan curve divides the plane into two regions, each of whose boundary is the curve itself. One region is unbounded, and called the exterior region and the other is bounded and called the interior region. Any path between the two regions must intersect the curve.*

We will not prove the theorem here, but show a quick proof that K_5 is not embeddable.

Suppose G were a plane curve isomorphic to K_5 and label the vertices $V(G) = \{v_1, \ldots v_5\}$. There is a subgraph defined by the cycle $C = v_1, v_2, v_3$. This cycle must form a Jordan curve. Thus v_4 must be either in the interior or exterior regions. Suppose it is in the interior region. Then We can consider the graph $G - v_5$, which consists of the cycle and some edges inside the cycle. If the vertex v_5 is in the exterior region to C, then the edge $v_4 v_5$ must intersect C, a contradiction. Otherwise, v_5 is inside one of the cycles $C_{12} = v_1, v_2, v_4$, $C_{13} = v_1, v_3, v_4$, or $C_{23} = v_2, v_3, v_4$. Say it is inside C_{12}. Then the edge $v_3 v_5$ must intersect C_{12}, a contradiction. Similar arguments deal with the remaining cases.

Although K_5 cannot be embedded in the plane, it can be embedded in the torus. Show picture. Also, $K_{3,3}$ can be embedded in a Moebius band. Note that the torus has Euler characteristic 0 and

We can now rethink the theorem that $p - q + r = 2$. This is true for plane graphs, but not for graphs embedded in the torus. The theorem of Euler is that if a graph divides the torus up into regions each of which is simply connected, then $p - q + r$ is the same no matter which graph. Simply connected means that any loop can be deformed into a point. Notice that there are regions of the torus for which this is not true.

4 Scheduling problem

Suppose you wish to assign times for final exams. You should schedule them so that no student has two exams scheduled at the same time. We can represent this as a graph, where the vertices are the courses and there is an edge if any student is taking both courses. We will assign colors to the vertices to indicate the time of the exam. Clearly, we want adjacent edges to have different colors. This is called a coloring of the graph.

Definition 13 *A coloring of a graph G is function $f : V(G) \to S$, for some set S, such that $f(v) \neq f(w)$ if $vw \in E(G)$. Often we will choose $S \subset \{1, 2, 3, \ldots\}$. An n-coloring is a coloring where S has n elments.*

Remark 14 *If G is a (p, q)-graph, then there is always a p-coloring. It is more interesting to find the smallest n for which G has an n-coloring.*

Definition 15 *The chromatic number $\chi(G)$ is the minimal n such that G has an n-coloring.*

Consider G_1 and G_2 on C-p. 204. We see that G_1 has a 5-coloring. However, clearly we can reduce this. We can find a 3-coloring. However, it does not have a 2-coloring. Graph G_2 has a 4-coloring as shown. But no 3-coloring. Thus $\chi(G_1) = 3$, $\chi(G_2) = 4$. We can also see that graph H on C-p. 205 has $\chi(H) = 4$.

Proposition 16 *The minimum number of exam periods is given by the chromatic number of the graph.*

Proof. The chromatic number is realized by a coloring of the graph, and on the coloring, no two classes which share a student have the same time slot (color). We just need to show that this is the minimal number of exam periods. Suppose we have an assignment of exam periods which has fewer time slots. Then these produce an n-coloring where $n < \chi(G)$, which is a contradiction. ■

Unfortunately, it is generally very difficult to compute a chromatic number. Here is a result.

Theorem 17 *Let $\Delta(G) = \max\{\deg(v) : v \in V(G)\}$. Then*

$$\chi(G) \leq 1 + \Delta(G).$$

Proof. Induct on the order of G. Suppose $p = 1$, then $\Delta(G) = 0$ and $\chi(G) = 1$. This is the base case. For the inductive step, we assume that $\chi(G) \leq 1 + \Delta(G)$ for any graph of order P or less. Let G be a graph of order $P+1$. Let v be a vertex of maximal degree. $G-v$ has a $(1 + \Delta(G - v))$-coloring. If $\deg(v) \leq \Delta(G - v)$, then we have $\Delta(G) = \Delta(G - v)$, and furthermore there is a free color to use to color v (since there are $\Delta(G - v) + 1$ colors available, but only $\Delta(G - v)$ vertices adjacent to v). If $\deg v > \Delta(G - v)$, then we can introduce a new color and color v with that color. ■

Note, that we actually proved something stronger:

Proposition 18 $\chi(G) \leq \min\{1 + \Delta(G), 2 + \Delta(G - v)\}$.

5 Four color theorem

Consider a map. We wish to color the map in such a way that adjacent countries have different colors. The map coloring problem is to find the minimal number of such colors. This can be translated into the question of finding the chromatic number of graphs representing the adjacency between countries. The four color theorem states that

Theorem 19 *For any planar graph G, $\chi(G) \leq 4$.*

In other words, we can always color a planar graph with 4 colors. This problem has an interesting history. There were a number of false proofs since the problem began in 1852, proposed by Francis Guthrie in 1858, and told to the mathematical community some years later by his brother. It was finally solved by Appel and Haken in 1976. Their proof required a large number of cases (nearly 2000) which were each checked with a computer. Subsequently, the number of cases has been reduced to around 600, but still too many for any person to check by hand. Whether this constituted a proof was extremely controversial in the mathematics community (and, in some instances, is still controversial).

We will prove that 5 colors is enough, which is much easier.

Lemma 20 *Every planar graph G contains a vertex v such that* $\deg v \leq 5$.

Proof. We may assume G has at least 7 vertices (since otherwise this is obvious). The sum of all of the degrees of the vertices is equal to $2q$. If every vertex has degree larger then or equal to 6, then $2q \geq 6p$. On the other hand, we proved that since G is planar, that $q \leq 3p - 6$. Thus we would have that $6p \leq 2q \leq 6p - 12$, a contradiction. ■

Theorem 21 *For any planar graph G, $\chi(G) \leq 5$.*

Proof. We induct on the order of the graph p. If $p = 1$, then the chromatic number is 1. Now suppose $\chi(G) \leq 5$ if $p \leq P$. Let G be a graph of order $P + 1$. By the lemma, there is a vertex v with $\deg v \leq 5$. By the inductive hypothesis, $G - v$ has a 5-coloring. If $\deg v < 4$, then we can easily color v to get a 5-coloring of G, so we may assume that $\deg v = 5$. We may number the vertices adjacent to v as v_1, v_2, v_3, v_4, v_5 in a cyclical ordering around v (in the plane graph representation of G). If all 5 colors are not represented among the neighbors of v, then we can produce a 5-coloring, so we may assume that v_1 has color 1, v_2 has color 2, etc. We will now consider paths in $G - v$ with only 2 colors (the colors must alternate because it is a coloring). First suppose there is no path from v_1 to v_3 with only colors 1 and 3. Consider all paths from v_1 with only colors 1 and 3 and call this graph H. H is a subgraph of $G - v$. We can switch all of the colors in H (1 becomes 3, 3 becomes 1), since any edge from a vertex in H to a vertex not in H must be between a vertex colored 1 or 3 (in H) and a vertex colored 2, 4,or 5 (not in H). Then v has no neighbor colored 1 and so we can color v with 1. However, if $v_3 \in H$, then switching the colors would not result in a free color. So, suppose $v_3 \in H$. Then there is a cycle in G given by P followed by v_3, v, v_1. This cycle has an interior region and an exterior region, and so v_2 is either in the interior region or the exterior region and v_4 is in the other. By the Jordan curve theorem, any path from v_2 to v_4 must cross this cycle, which means that there is no path from v_2 to v_4 with all vertices of color 2 or 4. Now procede as we did in the case there is no path from v_1 to v_3 with only colors 1 and 3. ■

There is an alternate proof. Proceed as before until we know we have v with degree 5. We claim that we can find 2 vertices adjacent to v which are not adjacent. If not, then there would be a subgraph of G with a subgraph isomorphic to K_5, which would mean there is an plane graph isomorphic to K_5. Let u, w be vertices adjacent to v but not adjacent to each other. We can form a graph from $G - v$ by identifying u and w, and that graph is planar (we can turn it into a plane graph by deforming u and w to v along uv and wv). Thus there is a 5-coloring for this graph, but that gives a 5-coloring on $G - v$ which has 4 or fewer colors adjacent to v.