Math 443/543 Graph Theory Notes 6: Digraphs, traffic, and tournaments

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September 29, 2008

1 Digraphs

Recall the definition of a digraph.

Definition 1 *A* directed graph*, or* digraph*, is a set together with an irreflexive relation. A digraph has* directed edges*, or* arcs*, given by ordered pairs* (u, v)*. We will denote the set of all arcs of the digraph* D *as* $E_+(D)$*. An arc* (u, v) *is* adjacent from u *and* adjacent to v.

2 Traffic problem

Suppose we have a small town for which we are planning a road system. We have the following limitations:

1. We can usually have all of the roads bidirectional since there is not much traffic.
2. However, on Saturdays that the football team plays at home, there is a great deal of traffic and so we would like to make the roads one-way.
3. We sometimes have to do repairs on roads, so we would like to be able to have a detour around any road segment so that drivers can still get from any one place to any other place.
4. We would like to be able to get from any one place to any other place even on Saturdays, when the roads are one-way.

We turn this problem into a graph in the following way. Let every road intersection be a vertex, and connect the vertices if there is a road between them. Limitation 1 says that usually we have a proper graph. Limitation 2 says that on Saturdays, we want to produce a digraph by assigning a direction to each edge of the graph. Limitation 3 says that we do not want to have any bridges in the graph. Limitation 4 says that the digraph is *strongly connected*, as defined:

Definition 2 *A v_0v_k-path in a digraph G is a list $P = v_0, e_1, v_1, \ldots, e_k, v_k$ such that $v_i \in V(G)$ and $e_i = (v_{i-1}, v_i) \in E_+(G)$ is a directed edge and such that the vertices and edges are distinct. A graph is* strongly connected *if for every vertices $u, v \in V(G)$ there is a uv-path and a vu-path.*

Definition 3 *A graph is* orientable *if it is possible to assign directions to each edge such that the induced digraph is strongly connected.*

Thus the traffic problem amounts to whether or not a graph is orientable. The solution is the following theorem.

Theorem 4 *A connected graph is orientable if and only if it has no bridges.*

Proof. First we show that a connected, orientable graph has no bridges. Suppose G is a connected graph with a bridge uv. Now suppose we have assigned directions to the edges to produce a digraph G_+. Certainly G_+ contains a uv-path or a vu-path consisting of just the edge (u, v) or (v, u) (depending on our choice of orientation of that edge). Without loss of generality, say $(u, v) \in E_+(G_+)$. Then there is no vu-path in G_+, since $G - uv$ is disconnected and so any path in G between u and v must contain the edge uv, which is oriented the wrong way. Thus G_+ is not orientable.

Now we show that if G has no bridges and is connected, then it is orientable. Since G has no bridges, every edge is contained in a cycle. The cycle can be oriented in an obvious way. Now suppose that there is an edge not in the cycle between two vertices in the cycle. We can assign those directions however we wish, and if the cycle contains all of the vertices of G, it is clear that G would be strongly connected. Now suppose that there is a vertex not in C. Then there must be an edge $v_iw \in E(G)$ such that w is not in C. This edge must lie on a cycle $C_1 = w_1, w_2, w_3, \ldots, w_k$, where $w_1 = w$ and $w_2 = v_i$. We direct the edge $(w, v_i) = (w_1, w_2)$ and then direct the edges (w_i, w_{i+1}) if they have not already been directed. Finally, any edges between vertices in C_1 and C but not in the cycle, we can assign the direction arbitrarily. We claim that this is a strongly connected subgraph. Given any vertex, there is a path to a point on the cycle C and also there is a path from any point on the cycle C to the vertex. Now we can continue with more cycles if all vertices are not contained in C and C_1. ■

Show picture like Fig 7.1.

3 Tournaments

Definition 5 *A tournament T is an orientation of a complete graph. Thus for every pair of vertices $v, w \in V(T)$ either (v, w) or (w, v), not both, is in $E_+(T)$.*

These correspond roughly to round-robin type tournaments, where no ties are allowed. One question is whether it is possible to rank the vertices in the tournament as $v_1, v_2, \ldots, v_p$ in such a way that $(v_i, v_{i+1}) \in E_+(T)$, where we think of $(v, w) \in E_+(T)$ means that v wins over w.

Definition 6 *The* indegree *of a vertex* v, *denoted* $i\deg(v)$, *is the number of vertices adjacent to* v. *The* outdegree *of* v, *denoted* $o\deg(v)$, *is the number of vertices adjacent from* v.

Definition 7 *The* length *of a path* P *is the number of arcs in* P. *The* distance $d(u,v)$ *from vertex* u *to vertex* v *the minimum length of paths from* u *to* v.

Note that $d(u,v)$ need not equal $d(v,u)$.

Theorem 8 *Let* T *be a tournament. Suppose* $v \in V(T)$ *is such that* v *has maximal outdegree. Then the distance from* v *to any other vertex in* $V(T)$ *is at most 2.*

Proof. Suppose the vertices adjacent from v are $\{v_1, \ldots, v_n\}$ and then the vertices adjacent to v are $\{u_1, \ldots, u_m\}$. We wish to show that for each vertex u_i, there is a vertex v_j adjacent to u_i. If there were a vertex u_i such that no vertex v_j is adjacent to u_i for all $j = 1, 2, \ldots, n$, then $o\deg(u_i) \geq n+1$, which would mean that v is not maximal, a contradiction. ■

The interpretation is this. If there is a tournament where every team plays every other tournament (a round-robin tournament), then the winner (the team with the most wins; there may be several) has only lost to teams which have lost to teams they have beaten.

Definition 9 *A* hamiltonian path *in a digraph is a path containing all of the vertices.*

Theorem 10 *Every tournament contains a hamiltonian path.*

Note that this means that we can rank teams in the tournament as suggested above. Note that this may not be unique, and also that the winner might not be what you expect!

Proof. We induct on the number of vertices p in the tournament. It is certainly true for $p = 1$, and we can easily check it for $p = 2, 3, 4$. Now suppose it is true for $p \leq n$. Let T be a tournament with n vertices. We note that $T - v$ is still a tournament (where $T - v$ removes the vertex v and all arcs to and from v). By the inductive hypothesis, $T - v$ has a hamiltonian path $P = v_1, v_2, \ldots, v_n$. If (v, v_1) or $(v_n, v) \in E_+(T)$, then T has a hamiltonian path. Otherwise, $(v, v_1) \in E_+(T)$ and there must be a minimal $k \leq n$ such that $(v_k, v) \in E_+(T)$. Then we have that $P' = v_1, \ldots, v_{k-1}, v, v_k, v_{k+1}, \ldots, v_n$ is a hamiltonian path. ■