Math 443/543 Graph Theory Notes 10: Pipelines, maximal and feasible flows on networks

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1 Pipeline problems

Suppose you have several pipelines arranged in a complicated way (with junctions and multiple input and output). Each pipe has a maximum capacity. We might ask:

- What is the maximum amount of stuff (oil, water, electricity, etc) that can be moved by the network from the inputs to the outputs?
- Is a certain collection of assigned inputs and outputs able to be attained by adjustments in the flow through the pipes?

This work is mostly from BM Chapter 7.

2 Networks and flows

We will recall some definitions for networks and then talk about flows.

Definition 1 *A network N is a digraph G together with a capacity function $c : E_+(G) \to [0, \infty]$ and two disjoint sets of vertices $X, Y \subset V(G)$. The vertices X are called the sources and the vertices Y are called the sinks. Vertices in $G - (X \cup Y)$ are called intermediate vertices and denoted as I.*

Definition 2 *We will consider functions f from the directed edges $E_+(G)$ to some set of numbers (usually positive real or positive integer. We denote*

$$f(K) = \sum_{e \in K} f(e)$$

if $K \subset E_+(G)$. Suppose $S \subset V(G)$. Let (S, S^c) denote the set of all directed edges from vertices in S to vertices in $S^c = V(G) - S$. We denote

$$f(S, S^c) = f^+(S)$$
$$f(S^c, S) = f^-(S).$$

In particular, $f^+(v)$ *is the sum of all values of* f *on arcs from* v *and* $f^-(v)$ *is the sum of all values of* f *on arcs to* v*. Also note that* $f^+(S) = f^-(S^c)$ *and* $f^-(S) = f^+(S^c)$*.*

Definition 3 *A flow through a network* N *is a function* $f : E_+(G) \to \mathbb{Z}_{\geq 0}$ *such that*

$$\begin{aligned} f(e) &\leq c(e) \text{ for all } e \in E_+(G) \\ f^-(v) &= f^+(v) \text{ if } v \in I. \end{aligned}$$

We think of f as specifying the amount of stuff flowing through a particular directed edge in the network. The first condition says we cannot exceed the capacity of any one pipe. The second is a conservation condition, saying that everything enters and leaves the network via X and Y.

Definition 4 *If* $S \subset V(G)$ *and* f *is a flow then we define the* resultant flow out of S relative to f *to be*

$$f^+(S) - f^-(S).$$

Similarly, the resultant flow into S relative to f *is*

$$f^-(S) - f^+(S).$$

The resultant flow tells how much net stuff leaves S (like a flux). Note the following:

Proposition 5 *For any* $S \subset V(G)$ *and flow* f*,*

$$f^+(S) - f^-(S) = \sum_{v \in S} \left[f^+(v) - f^-(v)\right].$$

Note that it is not true that

$$f^+(S) = \sum_{v \in S} f^+(v).$$

Proposition 6 *The resultant flow out of* X *is equal to the resultant flow into* Y*.*

Proof. We know that $f^+(v) = 0$ if $v \in I$, and so

$$\begin{aligned} f^+(X) - f^-(X) &= \sum_{v \in X} \left[f^+(v) - f^-(v)\right] \\ &= \sum_{v \in Y^c} \left[f^+(v) - f^-(v)\right] \\ &= f^+(Y^c) - f^-(Y^c) \\ &= f^-(Y) - f^+(Y). \end{aligned}$$

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Definition 7 *The* value of f *is defined as*

$$\operatorname{val} f = f^+(X) - f^-(X) = f^-(Y) - f^+(Y).$$

The value tells how much stuff is flowing through the network.

Definition 8 *A flow f on a network N is a* maximal flow *if there is no other flow on N with larger value.*

Thus a maximal flow is one which transmits the most stuff through the network.

Proposition 9 *For any network N, there is a new network N' such that $X' = \{x\}$, $Y' = \{y\}$, and there is a one-to-one correspondence of flows f on N and flows f' on N' such that*

$$\operatorname{val} f' = \operatorname{val} f.$$

Proof. Let N' be the network obtained from N by adding vertices x and y, arcs from x to each element of X and arcs from each element of Y to y. Give the new arcs capacity equal to infinity. Given a flow f' on N', there is an obvious subflow f on N. Given a flow f on N, we can construct the flow f' by setting

$$f'(a) = \begin{cases} f(a) & \text{if } a \in E_+(N) \\ f^+(v) - f^-(v) & \text{if } a = (x, v) \\ f^-(v) - f^+(v) & \text{if } a = (v, y) \end{cases}.$$

We see that $\operatorname{val} f' = \operatorname{val} f$. ∎

For this reason, we will often confine ourselves to networks with a single source x and a single sink y.

Definition 10 *Let N be a network with a single source x and a single sink y. A* cut *in N is a set (S, S^c) of arcs where $x \in S$ and $y \in S^c$.*

Consider Figure 1. This shows a flow. Notice that it is not maximal.

Definition 11 *The* capacity *of a cut K is equal to*

$$\operatorname{cap} K = \sum_{a \in K} c(a).$$

A minimum cut *is a cut K such that there is no cut K' with* $\operatorname{cap} K' < \operatorname{cap} K$.

A minimum cut is like the "weakest link" in the chain. If one could turn the network into a linear path from x to y, the minimum cut would be the smallest capacity in that chain. Notice the cut in Figure 1.

The key theorem about maximum flows and minimum cuts is the following.

Theorem 12 (Max Flow/Min Cut Theorem) *If f^* is the maximum flow and K_* is the minimum cut, then*

$$\operatorname{val} f^* = \operatorname{cap} K_*.$$

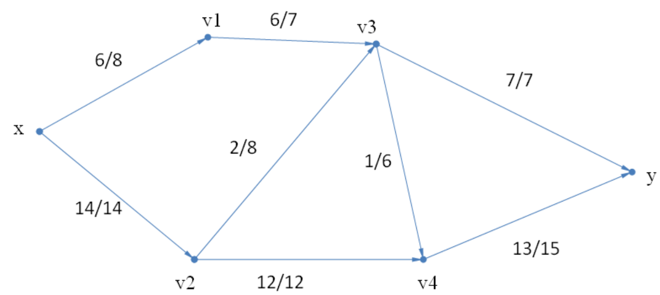

Figure 1: A flow

We will prove this soon, but first let's prove a more modest few things.

Lemma 13 *For any flow f and any cut (S, S^c) in N,*

$$\text{val}\, f = f^+(S) - f^-(S).$$

Proof. We know that

$$f^+(x) - f^-(x) = \text{val}\, f$$

and that

$$f^+(v) - f^-(v) = 0$$

for any $v \in S - x$. Thus we get that

$$\text{val}\, f = \sum_{v \in S} \left[f^+(v) - f^-(v) \right] = f^+(S) - f^-(S).$$

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Theorem 14 *For any flow f and any cut $K = (S, S^c)$ in N,*

$$\text{val}\, f \leq \text{cap}\, K.$$

Equality holds only if and only if $f(a) = c(a)$ for all $a \in (S, S^c)$ and if $f(a) = 0$ for all $a \in (S^c, S)$.

Corollary 15 *If f^* is the maximum flow and K_* is the minimum cut, then*

$$\operatorname{val} f^* \leq \operatorname{cap} K_*.$$

Note, we have proved one half of the Max Flow/Min Cut Theorem. The other inequality will be proven later.

Corollary 16 *If f is a flow and K is a cut such that* $\operatorname{val} f = \operatorname{cap} K$, *then f is a maximum flow and K is a minimum cut.*

Proof. We have that

$$\operatorname{val} f \leq \operatorname{val} f^* \leq \operatorname{cap} K_* \leq \operatorname{cap} K,$$

but the assumptions imply that these are all equalities. In particular, f is a maximum flow and K is a minimum cut. ∎

Corollary 17 *For any flow f and any cut $K = (S, S^c)$ in N, if $f(a) = c(a)$ for all $a \in (S, S^c)$ and if $f(a) = 0$ for all $a \in (S^c, S)$, then f is a maximum flow and K is a minimum cut.*

Proof of Theorem 14. We know that

$$\begin{aligned} f^+(S) &\leq \operatorname{cap} K \\ f^-(S) &\geq 0 \end{aligned}$$

so

$$\begin{aligned} \operatorname{val} f &= f^+(S) - f^-(S) \\ &\leq \operatorname{cap} K. \end{aligned}$$

The equality is if $f^+(S) = \operatorname{cap} K$ and $f^-(S) = 0$, so the second statement follows. ∎

3 Proof of Max Flow/Min Cut Theorem

In this section, we will consider the following types of paths (which are different from directed paths considered earlier).

Definition 18 *A $v_0 v_{k+1}$-*semipath *is a list $v_0, a_0, v_1, a_1, v_2, a_2, v_3, \ldots, a_k, v_{k+1}$ where v_i are vertices and a_i are arcs such that either $a_i = (v_i, v_{i+1})$ or $a_i = (v_{i+1}, v_i)$, and no vertex is repeated. Arcs of the first type are called* forward arcs *and arcs of the second type are called* reverse arcs.

We note that given a flow f on a network N together with a semipath P from x to y, we can produce a new flow $\tilde{f}$ by making

$$\tilde{f}(a) = \begin{cases} f(a) + \varepsilon & \text{if } a \text{ is a forward arc} \\ f(a) - \varepsilon & \text{if } a \text{ is a reverse arc} \\ f(a) & \text{otherwise} \end{cases},$$

as long as $f(a) + \varepsilon \leq c(a)$ and $f(a) - \varepsilon \geq 0$. The construction is designed to ensure that $f^+(v) = f^-(v)$ if $v \in I$.

We will now consider a way to use these semipaths to increase the value of a flow. For a xy-path P, define

$$\iota(a) = \begin{cases} c(a) - f(a) & \text{if } a \text{ is a forward arc in } P \\ f(a) & \text{if } a \text{ is a reverse arc in } P \end{cases}$$

and define

$$\iota(P) = \min_{a \in P} \iota(v).$$

Note that $\iota(a)$ is how much we can increase the forward flow or decrease the backward flow. We can now choose a new semipath

$$\hat{f}(a) = \begin{cases} f(a) + \iota(P) & \text{if } a \text{ is a forward arc} \\ f(a) - \iota(P) & \text{if } a \text{ is a reverse arc} \\ f(a) & \text{otherwise} \end{cases}.$$

Note that $\hat{f}$ is a new flow, since it satisfies the conditions to ensure $0 \leq \hat{f}(a) \leq c(a)$. Also note that

$$\operatorname{val} \hat{f} = \operatorname{val} f + \iota(P).$$

Theorem 19 *A flow f is a maximum flow if and only if N contains no xy-semipaths P with $\iota(P) > 0$.*

Proof. If N contains such a semipath P, we have shown how to increase the value of f, and so f is not a maximum. Now suppose N contains no such semipaths. We let S be the set of all vertices v such that there is a xv-semipath P_v such that $\iota(P_v) > 0$, together with x. We know that y is not in this set (by assumption), and so (S, S^c) is a cut. We will now show that each arc in (S, S^c) satisfies $f(a) = c(a)$ and every arc in (S^c, S) satisfies $f(a) = 0$. By Corollary, 17 this would imply that f is a maximum flow. Now suppose $a \in (S, S^c)$ and $a = (v, w)$. Then There is a xv-path P_v in N such that $\iota(P_v) > 0$. if $f(a) < c(a)$, then we could extend P_v to a xw-path, so we must have that $f(a) = c(a)$. Similarly, if we have $a \in (S^c, S)$ and $a = (w, v)$, then if $f(a) > 0$ then we could extend P_v to a xw-semipath. This completes the proof. ∎

Thus, in the process of the proof, we have shown that, given a flow, we can construct a maximum flow by incrementally considering xy-semipaths P with $\iota(P) > 0$ (these are called f-incremental paths in BM), finding new flows $\hat{f}$, and continuing until there are no such semipaths left. This flow will be a maximum and its value will be equal to the minimum cut, also shown in the proof. Thus, we have proven the Max Flow/Min Cut Theorem.