

FINAL

December 12th, 2008

Your Name: _____

Directions:

- a. You MAY use your book or your notes or a calculator.
- b. Please ask for extra scrap paper if needed.
- c. Show all work. Unless otherwise noted, a solution without work is worth nothing.
- d. Circle your answers.
- e. Good Luck!

Score:

1. _____

2. _____

3. _____

4. _____

5. _____

6. _____

7. _____

8. _____

Total _____/165

1. (10pts) Show that $K_{3,3}$ is not isomorphic to K_6 .

2. (20pts) Give an example of a nonplanar graph with a 2-coloring. Give an example of a planar graph that has no 2-coloring. Justify your answer.

3. (15pts)

Consider the following graph.



a. (5pts) Find a subgraph which is not isomorphic to an induced subgraph.

b. (5pts) Find a topological minor which is not isomorphic to a subgraph.

c. (5pts) Find a minor which is not isomorphic to a topological minor.

4. (25pts) Let C_n be the graph representing the cycle with n vertices, and let P_n be the graph representing a single path with n vertices. For each of the following graphs, state whether they are Eulerian, traversable, Hamiltonian, and/or 2-connected.

a. C_4

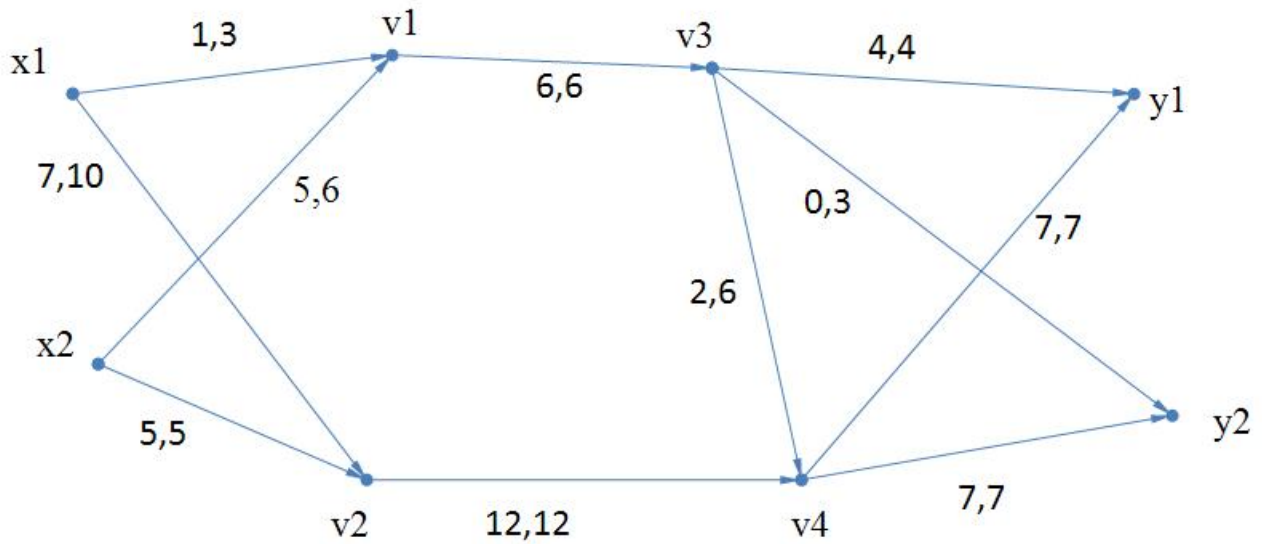
b. C_{20}

c. P_5

d. K_4

e. $K_{3,3}$

5. (35pts) Consider the following network. The first number is the flow through the edge, and the second number is the capacity of the edge. A maximum flow is given.

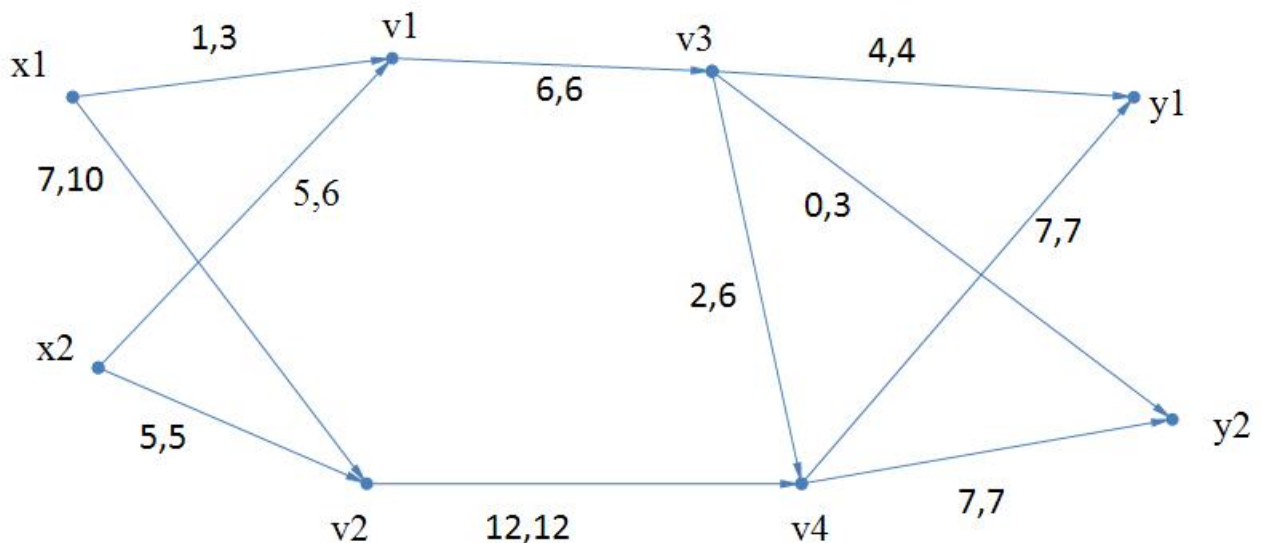


a. (20pts) Which of the following will allow us to increase the value of the flow?

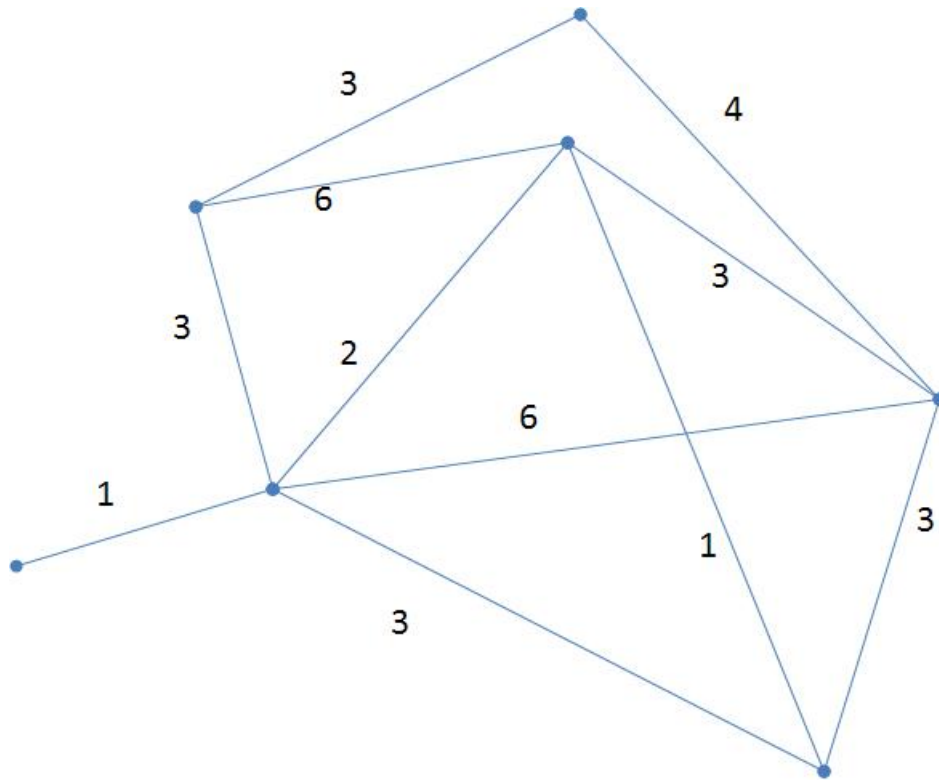
- Add an arc from v_1 to v_2 with capacity 4.
- Add an arc from v_1 to v_4 with capacity 1.
- Add an arc from v_2 to v_3 with capacity 2.
- Increase the capacity of the arc from v_2 to v_4 .

b. (15pts) Suppose we wish to find a feasible flow with supplies $\sigma(x_1) = 9, \sigma(x_2) = 12$ and demands $\delta(y_1) = 10, \delta(y_2) = 5$. Change the network to give a network with one source X and one sink Y such that if there is a maximal flow with value 15 then there is a feasible flow for the original network. Is there a feasible flow?

I have repeated the network below for your convenience.



6. (20pts) Find a minimal weight spanning tree of the following network.



7. (15pts) Suppose we have a strongly-connected web in which Page 1 is linked to from every other page and such that Page 1 links to itself. Now suppose we compute the PageRank directly from the link matrix (without adding the matrix of all $1/n$'s). Show that Page 1 must be ranked first by PageRank.

8. (25pts)

a. (10pts) Show that any tree with at least two vertices must have a vertex of degree 1. Hint: consider the sum of all of the degrees.

b. (10pts) Suppose I have a tree T_p with vertices v_1, \dots, v_p . Further, suppose that v_1 is a vertex of degree 1, and it is connected by an edge to vertex v_2 . Explain the difference between the Laplacian matrix for T_p and the Laplacian matrix for T_{p-1} , which is the tree $T_{p-1} = T_p - v_1$.

c. (5pts) Show directly that the tree T_2 with two vertices has $t(T_2) = 1$ (Do not use the result that $t(G)$ is the number of spanning tree is the number of spanning trees of G).

d. (Extra Credit: 10pts) Use the previous statements to show inductively that any tree T has $t(T) = 1$. (Do not use the result that $t(G)$ is the number of spanning tree is the number of spanning trees of G). Hint: you may use the fact that

$$\det \begin{pmatrix} a_{11} + r & a_{21} & a_{32} & \cdots & a_{n1} \\ a_{12} & a_{22} & \ddots & & a_{n2} \\ a_{13} & \ddots & \ddots & & \vdots \\ \vdots & & & & \vdots \\ a_{1n} & a_{2n} & \cdots & \cdots & a_{nn} \end{pmatrix} = \det A + r \det \hat{A}_{11}$$

where $A = (a_{ij})$ and \hat{A}_{11} is the matrix A with the 1st row and 1st column removed.