Problems on small world

November 19, 2012

1) A 3D grid is the graph consisting of vertices at points \((x, y, z)\), where each of these are integers, say all between 0 and \(n\), and two vertices are connected by an edge if ONE of their coordinates differ by 1 and the others are the same (so there is an edge between \((0, 0, 1)\) and \((0, 1, 0)\)). This is the obvious 3D “cube wireframe” analogy to the 2D grids we considered. Suppose we add in one random edge to each vertex \(v\) which goes to another vertex \(u\) with probability proportional to \(d(u, v)^{-q}\). Fix \(v\) and let \(R_d\) be the ring of vertices a distance at least \(d\) from \(v\) and at most \(2d\) from \(v\). What value should \(q\) have so that the probability that the random edge from \(v\) jumps into \(R\) can be bound by a number independent of \(d\) (and hence independent of the choice of ring \(R_d\))? Justify your answer.

2) Go to wikipedia.com and try to find a short path from "Lynah Faithful" to "Zona Zoo" using only links on each page. It may be helpful to know that "Lynah Faithful" refers to hockey fans at Cornell University in Ithaca, NY, and "Zona Zoo" refers to University of Arizona student fans. What is your strategy?

3) Consider the complete graph \(K_n\), the path with \(n\) vertices \(P_n\), and the cycle with \(n\) vertices. Compute the characteristic path length and the clustering coefficient.

4) On the 2D grid, show that there are no more than \(4i\) vertices a distance \(i\) from a given vertex, and no less than \(i + 1\) vertices a distance \(i\) from a given vertex.

5) Show that there are no more than \(2^{2i+2}\) vertices in \(B_i\).

In the next few questions, we see why our argument for decentralized search does not work for \(q = 2\). (Note that Kleinberg shows that you expect it to take longer, but we are only seeing why our argument did not show these have short paths.

6) Show the estimates:
   a)
   \[
   \sum_{w \neq v} d(w, v)^{-q} \leq c
   \]
   for some constant \(c\) if \(q > 2\) (hint: p-series converge for \(p > 1\)) and
   b)
   \[
   \sum_{w \neq v} d(w, v)^{-q} \leq \frac{1}{2 - q} (2n)^{2-q}
   \]
if $q < 2$. 

7) If $q > 2$, show that the probability $P_j$ that a message in stage $j$ (i.e., the message is currently in $B_{j+1}$ but not $B_j$) enters $B_j$ after one step satisfies 

$$P_j \geq \frac{1}{2(q-2)2^{q+1}c}$$

for the $c$ in problem 6.

8) If $q > 2$, show that our bound for the expected number of steps in $E_j$ is 

$$E_j \leq \sum_{i=1}^{\infty} \left( 1 - \frac{1}{2(q-2)2^{q+1}c} \right)^i \leq 2^{(q-2)j}2^{q+1}c - 1$$

and so the expected number of steps in the process is 

$$E \leq \sum_{j=1}^{\log_2 n} 2^{(q-2)j}2^{q+1}c$$

$$= 2^{q+1}c \sum_{j=1}^{\log_2 n} 2^{(q-2)j}$$

$$= 2^{q+1}c \left( \frac{2^{(q-2)(\log_2 n+1)} - 1}{2^{(q-2)} - 1} - 1 \right)$$

$$\leq \frac{2^{q+1}c2^{q-2}}{2^{(q-2)} - 1} n^{q-2}.$$ 

(Technically, this is only if $\log_2 n$ is an integer, but this estimate is good enough.)

9) If $q < 2$, show that the probability $P_j$ that a message in stage $j$ (i.e., the message is currently in $B_{j+1}$ but not $B_j$) enters $B_j$ after one step satisfies 

$$P_j \geq \frac{(2-q)2^{(2-q)j}}{2^{q+1}(2n)^{2-q}} = C \frac{2^{(2-q)j}}{n^{2-q}}$$

for appropriate $C$.

10) If $q < 2$, show that our bound for the expected number of steps in $E_j$ is 

$$E_j \leq \sum_{i=1}^{\infty} \left( 1 - C \frac{2^{(2-q)j}}{n^{2-q}} \right)^i \leq \frac{n^{2-q}}{C2^{(2-q)j}} - 1$$
and so the expected number of steps in the process is

\[
E \leq \sum_{j=1}^{\log_2 n} \frac{n^{2-q}}{C 2^{(2-q)j}}
\]

\[
\leq \frac{n^{2-q}}{C} \left[ \sum_{j=1}^{\infty} 2^{-(2-q)j} \right]
\]

\[
= \left( \frac{1}{C \cdot 1 - 2^{-(2-q)}} \right) n^{2-q}
\]

\[
= \left( \frac{1}{C \cdot 2^{(2-q)} - 1} \right) n^{2-q}.
\]