# 1 Economy trees

We wish to prove

**Theorem 1** *Let $(G, \phi)$ be a network. An economy tree $T_E$ (produced using Kruskal's algorithm) has minimal weight among spanning trees (i.e., it is a minimal spanning tree).*

We will step through the proof:

1) We can assume that there is a spanning tree $T_0$ of minimal weight (why?)

2) To show that, we will produce a sequence of trees starting with $T_E$ with nonincreasing weights. First order the edges in $T_E$ according to their weights, i.e., $E(T_E) = \{e_1, \ldots, e_{p-1}\}$ where $\phi(e_i) \leq \phi(e_{i+1})$. Let $e_j \in E(T_E)$ be the first edge that is not in $T_0$. Show that the graph $G_0 = T_0 + e_j$ must have a cycle $C$.

3) Show that the cycle must contain an edge $e_0$ that is an edge in $C$ but not in $T_E$ and so $e_0$ is an edge in $T_0$.

4) Consider the graph $T_0' = T_0 - e_0 + e_j$. Show that this must be a spanning tree.

5) Use the fact that $T_0$ has minimal weight to show that $\phi(e_0) \leq \phi(e_j)$.

6) Use the way the economy tree is constructed to argue that $\phi(e_j) \leq \phi(e_0)$, and so $\phi(e_j) = \phi(e_0)$ and $\phi(T_0') = \phi(T_0)$.

7) Conclude the proof by arguing that by starting this process with $T_0'$ instead of $T_0$ and proceeding several times, eventually the economy tree is produced, showing that $\phi(T_E) = \phi(T_0)$.

Compute the economy tree of the following network: