Recall Dijkstra's algorithm.

1. Let $\ell(u_0) = 0$ and let $\ell(v) = \infty$ for all $v \neq u_0$. Let $S_0 = \{u_0\}$ and let $i = 0$. We refer to i as the step.

2. For each $v \in S_i^c$, replace $\ell(v)$ with
$$\min_{u \in S_i} \{\ell(v), \ell(u) + \phi(uv)\}.$$

3. Compute M to be
$$M = \min_{v \in S_i^c} \{\ell(v)\}$$
and let u_{i+1} be the vertex which attains M.

4. Let $S_{i+1} = S_i \cup \{u_{i+1}\}$.

5. If $i = p-1$, stop. If $i < p-1$, then replace i with $i+1$ and goto step 2.

We will show that it finds the shortest path, that is, at the end $\ell(u) = d(u_0, u)$ for all vertices u. In fact, we prove something stronger: at each step i, $\ell(u) = d(u_0, u)$ for all $u \in S_i$. We will prove this by induction on i.

1) Show the base case $i = 0$ is true.

2) We now suppose the inductive hypothesis and need to prove that after step $i+1$, $\ell(u) = d(u_0, u)$ for all $u \in S_{i+1}$. In particular, we just need to show that after step $i+1$, $\ell(u_{i+1}) = d(u_0, u_{i+1})$. Let $P = v_1, \ldots, v_k$ be a minimal path from $v_1 = u_0$ to $v_k = u_{i+1}$, i.e., $\phi(P) = d(u_0, u_{i+1})$. Argue that $v_1, \ldots, v_j$ is a minimal path for any $j \leq k$.

3) Now show that if $v_{k-1} \in S_i$ (suppose this for problems 3,4,5) then $P' = v_1, \ldots, v_{k-1}$ is a minimal path and
$$\phi(P') = \ell(v_{k-1}).$$

4) Show that
$$d(u_0, u_{i+1}) = \phi(P) = \ell(v_{k-1}) + \phi(v_{k-1}u_{i+1})$$
and that
$$\ell(v_{k-1}) + \phi(v_{k-1}u_{i+1}) \geq \ell(u_{i+1})$$
using the steps in the algorithm.

5) Show that
$$d(u_0, u_{i+1}) \leq \ell(u_{i+1})$$
because $\ell(u_{i+1})$ corresponds to the length of some path. Conclude that $d(u_0, u_{i+1}) = \ell(u_{i+1})$.

6) We still need to show that $v_{k-1} \in S_i$. Let j be the smallest number such that $v_j \notin S_i$. Argue that
$$\ell(v_j) \leq \ell(v_{j-1}) + \phi(v_{j-1}v_j)$$

(using the algorithm) and argue that if P_j is the path $v_1, \ldots, v_j$

$$\ell(v_{j-1}) + \phi(v_{j-1}v_j) = \phi(P_j) \leq \phi(P)$$

and that

$$\phi(P) \leq \ell(u_{i+1}).$$

7) But since v_j is not in S_i, argue that $\ell(u_{i+1}) \leq \ell(v_k)$. Conclude that

$$\ell(v_j) = \ell(v_{j-1}) + \phi(v_{j-1}v_j) = \phi(P_j) = \phi(P) = \ell(u_{i+1})$$

by considering all of the above equalities. Therefore, $P_j = P$, so $j = k$ and $v_{k-1} \in S_i$.

Compute all shortest paths from v_0 in the following graph.