MATH 443/543

FINAL

Your Name: ____

Directions:

a. Do 5 of the 7 problems.
b. Be sure to attach the cover sheet to the front of your solutions.
c. The exam is due Thursday December 18 at 3pm. Please drop it off at the math homework drop off or Math 204.
d. You may use your notes or other resources, but not your classmates. Be sure to cite any works that you use.
e. Good Luck!

Score:

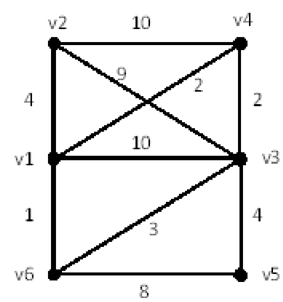
1.	____________
2.	____________
3.	____________
4.	____________
5.	____________
6.	____________
7.	____________
Total	____________

Problem 1. Let C_n be the graph representing the cycle with n vertices, and let P_n be the graph representing a single path with n vertices. For each of the following graphs, state whether they are Eulerian, traversable, Hamiltonian, and/or planar.

a. C_4
b. C_{20}
c. P_5
d. K_4
e. $K_{3,4}$

Problem 2. Recall Dijkstra's algorithm. Suppose that, in addition to keeping track of $\ell(v)$, we have another parameter $p(v)$ which keeps track of the previous vertex in a short path. While we run the algorithm, any time we replace $\ell(v)$ with $\ell(u) + \phi(uv)$, we set $p(v) = u$.

a. Illustrate how this determines the shortest paths to the base vertex by calculating all shortest paths to vertex v_2 in the following network:



b. Show how the collection of paths determined in part (a) form a spanning tree of the network above. Explain why (in any graph) the collection of paths thus determined by Dijkstra's algorithm must form a spanning tree.

c. Give a minimal spanning tree for the network above. Why is this different than the tree determined by Dijkstra's algorithm?

Problem 3. The Cartesian product of graphs G and H is the graph $G \times H$ such that $V(G \times H) = V(G) \times V(H)$ and there is an edge between (v, w) and (v', w') if and only if one of the following is true:

- $w = w'$ and $vv' \in E(G)$, or
- $v = v'$ and $ww' \in E(H)$.

a. Show that $K_2 \times K_2$ is isomorphic to C_4 (the cycle with 4 vertices).
b. The n-dimensional hypercube Q_n is defined inductively as $Q_1 = K_2$ and $Q_n = K_2 \times Q_{n-1}$. Use induction to show that Q_n has 2^n vertices and $n2^{n-1}$ edges. (Note: it may help to think

of the vertices as lying in $\mathbb{R}^n$ at the points $(x_1, \ldots, x_n)$ where each x_i is zero or one...consider this for Q_2 and Q_3. Feel free to look up hypercube and hypercube graph in wikipedia.)

c. Recall that an embedding of a graph with genus g into a surface of genus g satisfies Euler's formula:

$$V - E + F = 2 - 2g.$$

It can be shown that Q_n is bipartite (to keep this problem short enough, I do not ask you to prove this). Recall that a bipartite graph with at least 4 vertices satisfies $E \geq 2F$. Show that for $n \geq 2$ the genus g_n of Q_n is at least

$$1 + n2^{n-3} - 2^{n-1}.$$

(It is actually true that $g_n = 1 + n2^{n-3} - 2^{n-1}$, but we will not prove this.)

Problem 4. Suppose we have a network (G, Φ) where $\Phi : E \to \mathbb{R}_+$ gives *positive* weights to the edges. We number the vertices as $V(G) = \{v_1, \ldots, v_p\}$. The weighted Laplacian matrix is given by $L_\Phi = A - D$ where A is the weighted adjacency matrix $A_{ij} = \Phi(v_i v_j)$ if $v_i v_j \in E$ and 0 otherwise, and D is the diagonal matrix given by $D_{ii} = \sum_{j=1}^{p} A_{ij}$ and $D_{ij} = 0$ if $i \neq j$.

a. For which function Φ is L_Φ the usual graph Laplacian? Explain.

b. Show that for a function f on the vertices,

$$(L_\Phi f)_i = \sum_{\substack{j \text{ such that} \\ v_i v_j \in E}} \Phi(v_i v_j) \quad (f_j - f_i)$$

c. Show that $\det(-L_\Phi) = 0$ and that if λ is an eigenvalue of $-L_\Phi$ then $\lambda \geq 0$.

Problem 5. Suppose we have n boys, denoted $B_1, B_2, \ldots, B_n$, and n girls, denoted $G_1, G_2, \ldots, G_n$. Furthermore, suppose each girl has the exact same preference list, preferring B_1 first, then B_2, etc., so that each girl likes B_n the least. Also suppose that each boy has the exact same preference list, preferring G_1, then G_2, then G_3, etc.

a. Show that B_1 must be matched to G_1.

b. Use induction to show that B_n must be matched to G_n.

c. How many days would it take the mating ritual to find this matching?

Problem 6. Recall that the page rank was computed by looking at the eigenvector with eigenvalue one of the matrix $M = (1 - m) A + mS$, where A is the link matrix, S is the matrix of all $1/n$ (where n is the number of pages in the web), and m is a number between zero and 1. For this problem, you may assume that A is column stochastic.

Suppose the publisher of the rankings wants to boost the rankings of some of the pages (say, because those pages pay the publisher a fee). She can do this by changing the matrix S. For instance, suppose the web consists of 10 pages and the publisher wants to boost the ranking of the pages numbered 1 and 2. She can adjust rows 1 and 2 of the matrix S to boost the rankings of pages 1 and 2.

What should she do to the matrix S to boost those rankings while maintaining the fact that the rankings are well-defined (i.e., there is a unique ranking corresponding to the matrix M with the altered S matrix)? Explain why you think the rankings will be boosted and why the rankings are still well-defined. You may assume the web has no dangling nodes. (Note: you may want to use the probabilistic interpration of Page Rank to answer parts of this question).

Problem 7. A 3D grid is the graph consisting of vertices at points (x, y, z), where each of these are integers, say all between 0 and n, and two vertices are connected by an edge if ONE of their coordinates differ by 1 and the others are the same (so there is an edge between $(0, 0, 1)$ and $(0, 1, 1)$). This is the obvious 3D "cube wireframe" analogue of the 2D lattice grids we considered. Suppose we add in one random edge from each vertex v that goes to another vertex u with probability proportional to $d(u, v)^{-q}$. Fix v and let R_d be the ring of vertices a distance at least d from v and at most $2d$ from v. What value should q have so that the probability that the random edge from v jumps into R_d can be bounded by a number independent of d (and hence independent of the choice of ring R_d)? Justify your answer.