1 The 5-color theorem.

The main theorem is the following:

Theorem 1 *For any planar graph G, the chromatic number* $\chi(G) \leq 5$

First we prove a Lemma:

Lemma 2 *Every planar graph G contains a vertex v such that* $\deg v \leq 5$.

The proof is done in steps: Let G be a planar (p, q)-graph.

L1) We may assume that p is greater than or equal to 7. Why?

L2) Since the sum of the degrees is equal to $2q$ and if every vertex had degree greater than or equal to 6, show that $2q \geq 6p$.

L3) Recall that in a planar graph, $q \leq 3p - 6$ and deduce a contradiction if every vertex had degree greater than or equal to 6. This proves the lemma.

We are now ready to prove the theorem. Here is the outline:

We will induct on the order, p.

T1) The base case is $p = 1$. What is the chromatic number for a graph of order 1?

T2) Suppose $\chi(G) \leq 5$ for all graphs of order $p \leq P$. We will now consider a graph of order $P + 1$. By the Lemma, there is a vertex v with $\deg v \leq 5$. It follows that $G - v$ has a 5-coloring (i.e., $\chi(G - v) \leq 5$). Why?

T3) If $\deg v < 5$ then argue that G has a 5-coloring.

T4) Now suppose $\deg v = 5$. We can order the vertices adjacent to v in an embedding as v_1, v_2, v_3, v_4, v_5 and we can do this so that this is the order they appear going around the vertex. If all five colors are not represented in these, then argue that G has a 5-coloring.

T5) Now we can assume that each vertex has a different color, say v_1 has color 1, v_2 has color 2, etc. We will now consider the subgraph H of $G - v$ consisting of all paths from v_1 whose colors alternate between colors 1 and 3. Show that we can switch all the colors in H (so 1 becomes 3 and 3 becomes 1) and obtain a new coloring of $G - v$.

T6) If v_3 is not in H, then show that we can produce a coloring of G by switching the colors in H and then choosing the color of v appropriately.

T7) If v_3 is in H, argue there is a cycle in G given by a path in H combined with v. This cycle gives a Jordan curve.

T8) Use the Jordan Curve Theorem to argue that any path between v_2 and v_4 would have to intersect H.

T9) By changing the roles of $1, 3$ by $2, 4$, complete the proof of the theorem.

Here is an alternate way to prove the theorem:

T1-T3 are the same. Proceed as follows:

T4') If we can find two vertices u and w that are adjacent to v but not adjacent to each other, show there is a *planar* graph formed from $G - v$ by identifying the two vertices u and w. Hint: deform the whole plane graph by moving u and w along uv and wv.

T5') Show that a 5-coloring for this graph gives a 5-coloring for G.

T6') Complete the proof of the theorem by showing that we must be able to find such vertices u and w, otherwise there is a subgraph isomorphic to K_5 and G is not planar.

2 Higher genus (homework)

The goal of this section is to prove the following:

Theorem 3 *If G is a graph of genus $g > 0$, then we have*

$$\chi(G) \leq \frac{7 + \sqrt{1 + 48g}}{2}.$$

We can use the following generalization of Euler's Theorem:

Theorem 4 *If G is a connected (p, q)-graph of genus g, then if r is the number of regions in a embedding of G into the surface of genus g, then*

$$p - q + r = 2 - 2g.$$

1) Show that if G is a graph of genus g, then

$$q \leq 3(p + 2g - 2).$$

2) Show that if G is a graph of genus g, then if v is a vertex in G with the smallest degree and $\delta = \deg v$, then

$$\frac{\delta}{2} p \leq q$$

3) It follows that

$$\frac{\delta}{2} p \leq 3(p + 2g - 2).$$

Furthermore, since $p \geq \delta + 1$, show that if $\delta \geq 6$,

$$\delta \leq \frac{5 + \sqrt{1 + 48g}}{2}.$$

4) Now show that the chromatic number for a graph of genus $g > 0$ is less than

$$\frac{7 + \sqrt{1 + 48g}}{2}$$

by considering $G - v$, where $\deg v \leq \frac{5+\sqrt{1+48g}}{2}$.

5) Use the theorem to show that the genus of K_8 is at least 2 and the genus of K_9 is at least 3.