1 The 5-color theorem.

The main theorem is the following:

**Theorem 1** For any planar graph $G$, the chromatic number $\chi(G) \leq 5$

First we prove a Lemma:

**Lemma 2** Every planar graph $G$ contains a vertex $v$ such that $\deg v \leq 5$.

The proof is done in steps: Let $G$ be a planar $(p, q)$-graph.

L1) We may assume that $p$ is greater than or equal to 7. Why?

L2) Since the sum of the degrees is equal to $2q$ and if every vertex had degree greater than or equal to 6, show that $2q \geq 6p$.

L3) Recall that in a planar graph, $q \leq 3p - 6$ and deduce a contradiction if every vertex had degree greater than or equal to 6. This proves the lemma.

We are now ready to prove the theorem. Here is the outline:

We will induct on the order, $p$.

T1) The base case is $p = 1$. What is the chromatic number for a graph of order 1?

T2) Suppose $\chi(G) \leq 5$ for all graphs of order $p \leq P$. We will now consider a graph of order $P + 1$. By the Lemma, there is a vertex $v$ with $\deg v \leq 5$. It follows that $G - v$ has a 5-coloring (i.e., $\chi(G - v) \leq 5$). Why?

T3) If $\deg v < 5$ then argue that $G$ has a 5-coloring.

T4) Now suppose $\deg v = 5$. We can order the vertices adjacent to $v$ in an embedding as $v_1, v_2, v_3, v_4, v_5$ and we can do this so that this is the order they appear going around the vertex. If all five colors are not represented in these, then argue that $G$ has a 5-coloring.

T5) Now we can assume that each vertex has a different color, say $v_1$ has color 1, $v_2$ has color 2, etc. We will now consider the subgraph $H$ of $G - v$ consisting of all paths from $v_1$ whose colors alternate between colors 1 and 3. Show that we can switch all the colors in $H$ (so 1 becomes 3 and 3 becomes 1) and obtain a new coloring of $G - v$.

T6) If $v_3$ is not in $H$, then show that we can produce a coloring of $G$ by switching the colors in $H$ and then choosing the color of $v$ appropriately.

T7) If $v_3$ is in $H$, argue there is a cycle in $G$ given by a path in $H$ combined with $v$. This cycle gives a Jordan curve.

T8) Use the Jordan Curve Theorem to argue that any path between $v_2$ and $v_4$ would have to intersect $H$.

T9) By changing the roles of 1, 3 by 2, 4, complete the proof of the theorem.

Here is an alternate way to prove the theorem:

T1-T3 are the same. Proceed as follows:

T4') If we can find two vertices $u$ and $w$ that are adjacent to $v$ but not adjacent to each other, show there is a planar graph formed from $G - v$ by
identifying the two vertices \( u \) and \( w \). Hint: deform the whole plane graph by moving \( u \) and \( w \) along \( uv \) and \( wv \).

T5’) Show that a 5-coloring for this graph gives a 5-coloring for \( G \).

T6’) Complete the proof of the theorem by showing that we must be able to find such vertices \( u \) and \( w \), otherwise there is a subgraph isomorphic to \( K_5 \) and \( G \) is not planar.

2 Higher genus (homework)

The goal of this section is to prove the following:

**Theorem 3** If \( G \) is a graph of genus \( g > 0 \), then we have

\[
\chi(G) \leq \frac{7 + \sqrt{1 + 48g}}{2}.
\]

We can use the following generalization of Euler’s Theorem:

**Theorem 4** If \( G \) is a connected \((p, q)\)-graph of genus \( g \), then if \( r \) is the number of regions in a embedding of \( G \) into the surface of genus \( g \), then

\[
p - q + r = 2 - 2g.
\]

1) Show that if \( G \) is a graph of genus \( g \), then

\[
q \leq 3(p + 2g - 2).
\]

2) Show that if \( G \) is a graph of genus \( g \), then if \( v \) is a vertex in \( G \) with the smallest degree and \( \delta = \deg v \), then

\[
\frac{\delta}{2}p \leq q.
\]

3) It follows that

\[
\frac{\delta}{2}p \leq 3(p + 2g - 2).
\]

Furthermore, since \( p \geq \delta + 1 \), show that if \( \delta \geq 6 \),

\[
\delta \leq \frac{5 + \sqrt{1 + 48g}}{2}.
\]

4) Now show that the chromatic number for a graph of genus \( g > 0 \) is less than

\[
\frac{7 + \sqrt{1 + 48g}}{2}
\]

by considering \( G - v \), where \( \deg v \leq \frac{5 + \sqrt{1 + 48g}}{2} \).

5) Use the theorem to show that the genus of \( K_8 \) is at least 2 and the genus of \( K_9 \) is at least 3.