Math 443/543 Laplacian problems sheet 1

October 30, 2014

1) Find the eigenvalues of $-L$ for the graph $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_3v_4\}$. Note: two eigenvalues should be easy to find using our theorems. You will still need to find two more eigenvalues either by guessing eigenvectors (which is not too hard) or by using the fact that the eigenvalues are the solution to $\det(A - \lambda I) = 0$. The fact that the sum of the eigenvalues is equal to the trace may also be helpful.

2) Use the Matrix Tree Theorem to compute the number of spanning trees of $K_{1,m}$ (the answer should be fairly obvious, but I want to see that this number actually equals $t(G) = \det\left(-\hat{L}_{11}\right)$).

3) Compute the number of spanning trees of $K_{3,3}$ and compare to the result using the Matrix Tree Theorem (use matlab or wolfram alpha or some other tool).

4) Let $Q \in \mathbb{R}^{p\times q}$ be the directed edge-vertex adjacency matrix, $\hat{Q}$ be the matrix Q with the first row removed, and S be a subset of the columns representing edges forming a spanning tree in the graph as in the proof of the Matrix Tree Theorem, show that $\det \hat{Q}_S = \pm 1$. Hint: induct on p using the fact that a tree must have at least two vertices of degree one.

Alternative formulation. Let $Q \in \mathbb{R}^{p\times(p-1)}$ be a directed edge-vertex adjacency matrix corresponding to a tree. Let $\hat{Q}$ be the $\mathbb{R}^{(p-1)\times(p-1)}$ matrix gotten by removing the first row from Q. Show that $\det \hat{Q} = \pm 1$. Hint: induct on p using the fact that a tree must have at least two vertices of degree one.

5) Suppose we have a network (G, Φ) where $\Phi : E \to \mathbb{R}_+$ give positive weights to the edges. The weighted Laplacian matrix is given by $L_\Phi = A - D$ where A is the weighted adjacency matrix $A_{ij} = \Phi(v_iv_j)$ if $v_iv_j \in E$ and 0 otherwise, and D is the diagonal matrix given by $D_{ii} = \sum_{j=1}^{p} A_{ij}$.

A) Show that for a function f on the vertices,

$$(L_\Phi f)_i = \sum_{\substack{j \text{ such that} \\ v_iv_j \in E}} \Phi(v_iv_j) \quad (f_j - f_i)$$

B) Show that $\det(-L_\Phi) = 0$ and that if λ is an eigenvalue of $-L_\Phi$ then $\lambda \geq 0$.

6) Give an interpretation of the weighted Laplacian in terms of electrical networks and Kirchoff's law. What do the weights represent?