1 Basic Matrices

Recall a matrix is a collection of numbers, denoted as $A = (a_{ij})$, where $1 \leq i \leq m$ and $1 \leq j \leq n$. The matrix is said to be an $m \times n$ ("m by n") matrix. Consider the following $2 \times 3$ matrix

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$ 

If the entries of $B$ are denoted $b_{ij}$, we see that $b_{11} = 1$, $b_{12} = 2$, $b_{13} = 3$, $b_{21} = 4$, $b_{22} = 5$, $b_{23} = 6$.

The numbers $m$ and $n$ are called the dimensions. Matrices have rows and columns. The $i$th row is the matrix $[a_{i1}a_{i2}\cdots a_{in}]$. The $j$th column is the matrix

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$ 

If $m = n$ then we say the matrix is square. The entries $a_{ii}$ are said to be on the diagonal. The following $3 \times 3$ matrix has 1, 6, and 11 on the diagonal:

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix}.$$ 

2 Matrix multiplication

Before doing matrix multiplication, note that we can multiply a real number $a$ times a matrix $M$ to get a matrix $aM$ which consists of the same entries of $M$ all multiplied by $a$. For instance,

$$2 \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 10 & 12 & 14 \end{bmatrix}.$$ 

We can multiply a $m \times n$ matrix $A$ and a $n \times p$ matrix $B$ in the following way. Then there is a $m \times p$ matrix $C = AB$ (note that the order is important; $BA$ may not have a meaning, and even if it does, it may not equal $AB$) with entries

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$ 

Have a look again at the $m, n, p$ in the above description. It is important that for two matrices to multiply, they must have the appropriate dimensions. Here is an example of matrix multiplication:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 38 & 44 & 50 & 56 \\ 83 & 98 & 113 & 128 \end{bmatrix}.$$
Matrix multiplication is linear, in the sense that if \(A, B, M, N\) are appropriate dimensional matrices and \(a\) is a real number, \((A + aB)M = AM + a(BM)\) and \(A(M + aN) = AM + a(AN)\).

The identity matrix of dimension \(n\) is the \(n \times n\) matrix with 1 on the diagonal and 0 off the diagonal, and is often denoted as \(I\) or \(I_n\). For instance,

\[
I_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

The identity matrix \(I\) has the property that for any matrices \(A\) and \(B\), \(AI = A\) and \(IB = B\) if \(I\) is the appropriate dimension so the multiplication makes sense. In this sense, vectors are \(1 \times n\) matrices and so matrices act on vectors like \(u = Av\), where if \(A\) is a \(n \times m\) matrix, then \(v\) is a vector in \(\mathbb{R}^m\) and \(u\) is a vector in \(\mathbb{R}^n\).

The transpose of a matrix \(A\) switches the rows and columns and is denoted as \(A^T\). That is, if \(A = (a_{ij})\) is a \(m \times n\) matrix, then \(A^T = (b_{ij})\) is the \(n \times m\) matrix given by \(b_{ij} = a_{ji}\). We see that

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12
\end{bmatrix}^T = \begin{bmatrix}
1 & 5 & 9 \\
2 & 6 & 10 \\
3 & 7 & 11 \\
4 & 8 & 12
\end{bmatrix}.
\]

Note that if we consider a vector \(v\) to be a \(1 \times n\) matrix, then \(v^Tv\) is the usual dot product. A matrix \(A\) is symmetric if \(A^T = A\).

A permutation matrix is a matrix that is gotten from the identity by interchanging some of the columns. For instance, here is a permutation matrix

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

A permutation matrix \(P\) has the property that \(P^TP = I\). It also has the property that \(AP\) is the matrix obtained from \(A\) by switching the columns in the same way that was done to get from \(I\) to \(P\).

### 3 Eigenvalues and Eigenvectors

Given a square matrix \(M\), a vector \(v\), and a number \(\lambda\) (possibly complex), we say \(\lambda\) is an eigenvalue of \(M\) with corresponding eigenvector \(v\) if

\[
Mv = \lambda v.
\]

The collection of all vectors with the same eigenvalue is called the corresponding eigenspace. Eigenvectors are zeroes of the characteristic polynomial, \(\det (M - \lambda I)\).
(see determinants below). Since every polynomial can be factored into linear
terms over the complex numbers, we always have set of complex eigenvalues. A
very important theorem is the spectral theorem:

**Theorem 1** If \( M \) is a symmetric matrix, then all eigenvalues of \( M \) are real
and there is a matrix \( A \) such that \( A^T A = I \) and such that

\[
A^T MA = D
\]

where \( D \) is a matrix with the eigenvalues on the diagonal and zeroes elsewhere.

4 Nullspace and nullity

The *nullspace* of a matrix \( A \) is the set of vectors \( v \) such that \( Av = 0 \). It is thus
the eigenspace of the eigenvalue 0 if \( A \) is a symmetric matrix. The nullspace
is a vector space, meaning that for any two vectors \( v, w \) in the nullspace and
any real numbers \( a \) and \( b \), \( av + bw \) is in the nullspace. This follows because if
\( Av = 0 \) and \( Aw = 0 \), then

\[
A (av + bw) = a (Av) + b (Aw) = 0.
\]

A *linear combination* of vectors \( v_1, \ldots, v_k \) is a vector such that there exist
real numbers \( a_1, \ldots, a_k \) such that the vector can be expressed as

\[
a_1 v_1 + \cdots + a_k v_k.
\]

The *nullity* is the smallest number of vectors such that any vector in the
nullspace can be expressed as a linear combination of those vectors.

A square matrix \( A \) is invertible if there is another matrix, denoted \( A^{-1} \), such
that \( AA^{-1} = A^{-1} A = I \). A square matrix is invertible if and only if the nullity
is zero and if and only if its determinant is nonzero.

5 Determinants

The determinant of a square matrix can be defined inductively as \( \det [a] = a \)
for a \( 1 \times 1 \) matrix and then the determinant of a \( n \times n \) matrix is gotten as

\[
\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \hat{A}_{ij}
\]

for any \( i \), where \( \hat{A}_{ij} \) is the matrix with the \( i \)th row and \( j \)th column removed. This
is called expanding in the \( i \)th row. It is not hard to see that \( \det A = \det A^T \) and
so we can also expand in a column instead of a row. The determinant also has
the property that \( \det (AB) = (\det A) (\det B) \). It follows that \( \det (A^{-1}) = \frac{1}{\det A} \).
6 Systems of equations

A system of linear equations can be written as a matrix equation $Ax = b$ as follows. If $A = (a_{ij})$ and $x = (x_1, \ldots, x_n)$ and $b = (b_1, \ldots, b_m)$ then the matrix equation $Ax = b$ corresponds to the system

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
  &\vdots \\
  a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{align*}
\]