1 Basic Matrices

Recall a matrix is a collection of numbers, denoted as $A = (a_{ij})$, where $1 \leq i \leq m$ and $1 \leq j \leq n$. The matrix is said to be an $m \times n$ ("m by n") matrix. Consider the following 2×3 matrix

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

If the entries of B are denoted b_{ij}, we see that $b_{11} = 1$, $b_{12} = 2$, $b_{13} = 3$, $b_{21} = 4$, $b_{22} = 5$, $b_{23} = 6$.

The numbers m and n are called the *dimensions*. Matrices have rows and columns. The ith row is the matrix $[a_{i1}a_{i2} \cdots a_{in}]$. The jth column is the matrix

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

If $m = n$ then we say the matrix is *square*. The entries a_{ii} are said to be on the diagonal. The following 3×3 matrix has 1, 6, and 11 on the diagonal:

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix}.$$

2 Matrix multiplication

Before doing matrix multiplication, note that we can multiply a real number a times a matrix M to get a matrix aM which consists of the same entries of M all multiplied by a. For instance,

$$2 \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix} = 2 \begin{bmatrix} 2 & 4 & 6 \\ 10 & 12 & 14 \\ 18 & 20 & 22 \end{bmatrix}.$$

We can multiply a $m \times n$ matrix A and a $n \times p$ matrix B in the following way. Then there is a $m \times p$ matrix $C = AB$ (note that the order is important; BA may not have a meaning, and even if it does, it may not equal AB) with entries

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Have a look again at the m, n, p in the above description. It is important that for two matrices to multiply, they must have the appropriate dimensions. Here is an example of matrix multiplication:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 38 & 44 & 50 & 56 \\ 83 & 98 & 113 & 128 \end{bmatrix}.$$

Note the dimensions.

Matrix multiplication is linear, in the sense that if A, B, M, N are appropriate dimenional matrices and a is a real number, $(A + aB)\,M = AM + a\,(BM)$ and $A\,(M + aN) = AM + a\,(AN)$.

The *identity matrix* of dimension n is the $n \times n$ matrix with 1 on the diagonal and 0 off the diagonal, and is often denoted as I or I_n. For instance,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The identity matrix I has the property that for any matrices A and B, $AI = A$ and $IB = B$ if I is the appropriate dimension so the multiplication makes sense. In this sense, vectors are $1 \times n$ matrices and so matrices act on vectors like $u = Av$, where if A is a $n \times m$ matrix, then v is a vector in $\mathbb{R}^m$ and u is a vector in $\mathbb{R}^n$.

The *transpose* of a matrix A switches the rows and columns and is denoted as A^T. That is, if $A = (a_{ij})$ is a $m \times n$ matrix, then $A^T = (b_{ij})$ is the $n \times m$ matrix given by $b_{ij} = a_{ji}$. We see that

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}.$$

Note that if we consider a vector v to be a $1 \times n$ matrix, then $v^T v$ is the usual dot product. A matrix A is *symmetric* if $A^T = A$.

A *permutation matrix* is a matrix that is gotten from the identity by interchanging some of the columns. For instance, here is a permutation matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A permutation matrix P has the property that $P^T P = I$. It also has the property that AP is the matrix obtained from A by switching the columns in the same way that was done to get from I to P.

3 Eigenvalues and Eigenvectors

Given a square matrix M, a vector v, and a number λ (possibly complex), we say λ is an *eigenvalue* of M with corresponding *eigenvector* v if

$$Mv = \lambda v.$$

The collection of all vectors with the same eigenvalue is called the corresponding *eigenspace*. Eigenvectors are zeroes of the characteristic polynomial, $\det\,(M - \lambda I)$

(see determinants below). Since every polynomial can be factored into linear terms over the complex numbers, we always have set of complex eigenvalues. A very important theorem is the spectral theorem:

Theorem 1 *If M is a symmetric matrix, then all eigenvalues of M are real and there is a matrix A such that $A^T A = I$ and such that*

$$A^T M A = D$$

where D is a matrix with the eigenvalues on the diagonal and zeroes elsewhere.

4 Nullspace and nullity

The *nullspace* of a matrix A is the set of vectors v such that $Av = 0$. It is thus the eigenspace of the eigenvalue 0 if A is a symmetric matrix. The nullspace is a vector space, meaning that for any two vectors v, w in the nullspace and any real numbers a and b, $av + bw$ is in the nullspace. This follows because if $Av = 0$ and $Aw = 0$, then

$$A(av + bw) = a(Av) + b(Aw) = 0.$$

A *linear combination* of vectors $v_1, \ldots, v_k$ is a vector such that there exist real numbers $a_1, \ldots, a_k$ such that the vector can be expressed as

$$a_1 v_1 + \cdots + a_k v_k.$$

The *nullity* is the smallest number of vectors such that any vector in the nullspace can be expressed as a linear combination of those vectors.

A square matrix A is invertible if there is another matrix, denoted A^{-1}, such that $AA^{-1} = A^{-1}A = I$. A square matrix is invertible if and only if the nullity is zero and if and only if its determinant is nonzero.

5 Determinants

The determinant of a square matrix can be defined inductively as $\det[a] = a$ for a 1×1 matrix and then the determinant of a $n \times n$ matrix is gotten as

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \hat{A}_{ij}$$

for any i, where $\hat{A}_{ij}$ is the matrix with the ith row and jth column removed. This is called expanding in the ith row. It is not hard to see that $\det A = \det A^T$ and so we can also expand in a column instead of a row. The determinant also has the property that $\det(AB) = (\det A)(\det B)$. It follows that $\det\left(A^{-1}\right) = \frac{1}{\det A}$.

6 Systems of equations

A system of linear equations can be written as a matrix equation $Ax = b$ as follows. If $A = (a_{ij})$ and $x = (x_1, \dots x_n)$ and $b = (b_1, \dots, b_m)$ then the matrix equation $Ax = b$ corresponds to the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n &= b_m. \end{aligned}$$