1 Proof of the Matrix Tree Theorem

In order to prove the Matrix Tree Theorem, we need another characterization of the Laplacian.

Definition 1 *Let G be a directed (p,q)-graph. The oriented vertex-edge incidence graph is a $p \times q$ matrix $Q = [q_{ir}]$, such that $q_{ir} = 1$ if $e_r = (v_i, v_j)$ for some j and $q_{ir} = -1$ if $e_r = (v_j, v_i)$ for some j.*

Definition 2 *An orientation of a graph is a choice of direction for each edge, resulting in a directed graph.*

We will first show the following:

Proposition 3 *The Laplacian matrix L satisfies*

$$-L = QQ^T$$

for any oriented vertex-edge incidence graph (so, given an undirected graph, we can take any orientation), i.e.,

$$L_{ij} = -\sum_{r=1}^{q} q_{ir} q_{jr}.$$

P1) If $i = j$, then we see that $L_{ii} = -\deg v_i$. Note that q_{ir} is nonzero if v_i is in edge e_r. Show that

$$\sum_{r=1}^{q} (q_{ir})^2 = \deg v_i.$$

P2) Now for $i \neq j$, show that $q_{ir} q_{jr} = -1$ if $e_r = v_i v_j$, giving the result.

Remark 4 *This observation can be used to give a different proof that L_{ij} has all nonpositive eigenvalues.*

The proof of the matrix tree theorem proceeds as follows. We will compute

$$\det\left(-\hat{L}_{11}\right).$$

A property of determinants allows us to use the fact that $-L = QQ^T$ to compute this determinant in terms of determinants of submatrices of Q.

Proposition 5 (Binet-Cauchy Formula) *Let A be an $n \times m$ matrix and B be an $n \times m$ matrix (usually we think $n < m$). We can compute*

$$\det(AB) = \sum_S \det A_S \det B_S^T$$

where A_S is matrix consisting of the columns of A specified by S, and S ranges over all choices of n columns.

Note: We will not prove this theorem, but it is quite geometric in the case that $B = A^T$, which is our case. If we consider the rows of A, then $\det AA^T$ is equal to the square of the volume of the parallelopiped determines by these rows (since we can rotate A to be in $\mathbb{R}^n$). The formula says that the volume squared is equal to the sum of the squares of the volumes when projected onto the coordinate planes. These are generalizations of the pythagorean theorem!

We now begin the proof of the Matrix Tree Theorem. Recall that we have

$$-L = QQ^T$$

for a choice of orientation.

MT1) Show that

$$\begin{aligned} t(G) &= \det\left(-\hat{L}_{11}\right) = \det\left(\hat{Q}\hat{Q}^T\right) \\ &= \sum_S \left(\det \hat{Q}_S\right)^2 \end{aligned}$$

where $\hat{Q}$ is Q with the first row removed and S ranges over collections of $p-1$ edges in G.

MT2) Show that if S forms a disconnected graph (assume this for MT2,MT3), then $\hat{Q}_S$ looks like

$$\left(\begin{array}{cc} M & 0 \\ 0 & M' \end{array}\right).$$

MT3) Since M' consists of columns with exactly one 1 and one -1, show that $(0, 0, \ldots, 0, 1, 1, 1, \ldots 1)^T$ (respecting the form above) is a zero eigenvector, and that it follows that $\det \hat{Q}_S = 0$.

MT4) Return to general S. Show that if S contains a cycle, then since it has $p-1$ edges, the corresponding graph must be disconnected, and so $\det \hat{Q}_S = 0$.

MT5) Let $Q \in \mathbb{R}^{p\times(p-1)}$ be a directed edge-vertex adjacency matrix corresponding to a tree. Let $\hat{Q}$ be the $\mathbb{R}^{(p-1)\times(p-1)}$ matrix gotten by removing the first row from Q. Show that $\det \hat{Q} = \pm 1$. Hint: induct on p using the fact that a tree must have at least two vertices of degree one.

MT6) Show that it follows that

$$t(G) = \sum_S \left(\det \hat{Q}_S\right)^2$$

is equal to the number of spanning trees.