# Wagner's Theorem problems.

Let's revisit some definitions. Let $G = (V, E)$ be a graph.

**Definition 1** Removing a vertex *means removing that vertex from the vertex set of* $G$ *and removing all edges incident with that vertex from the edge set. We denote the graph obtained from* $G$ *by removing a vertex* $v$ *by* $G - v$*.*

**Definition 2** Removing an edge *means removing that edge from the edge set. We denote the graph obtained from* $G$ *by removing an edge* $uv$ *by* $G - uv$*.*

**Definition 3** Contracting an edge $e = uv$ *means removing* $u$ *and* $v$ *from the vertex set and replacing it by a new vertex* $z$ *and adjusting the edges edges such that* $z$ *is adjacent to all vertices that were adjacent to* $u$ *or* $v$*.*

**Definition 4** *A graph* $H$ *is a* minor *of a graph* $G$ *if* $H$ *can be obtained from* $G$ *by any combination of the following operations:*

- Removing vertices.
- Removing edges
- Contracting edges.

In this problem, we will discuss parts of the fact that Kuratowski's theorem is equivalent to the following theorem:

**Theorem 5 (Wagner's Theorem)** *A graph is planar if and only if it does not have a minor isomorphic to* $K_{3,3}$ *or* $K_5$*.*

Compare to Kuratowski's Theorem:

**Theorem 6 (Kuratowski's Theorem)** *A graph is planar if and only if it does not have a subgraph isomorphic to a subdivision of* $K_{3,3}$ *or* $K_5$*.*

- Question 1: Consider the Petersen graph in C-9, Problem 13. Show that it is not planar using both Wagner's Theorem and Kuratowski's Theorem.
- Question 2: Show that if a graph has a subgraph isomorphic to a subdivision of $K_{3,3}$ or $K_5$, then it has a minor isomorphic to $K_{3,3}$ or $K_5$.
- Question 3: Show that if a graph has a minor isomorphic to $K_{3,3}$, then it has a subgraph isomorphic to a subdivision of $K_{3,3}$.

The hard part is to show that if a graph has a minor isomorphic to $K_5$, then it has a subgraph isomorphic to a subdivision of $K_5$ or $K_{3,3}$. We will not prove this.