Survivable Network Design with Vertex and Edge Connectivity Constraints

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Background

- Survivable network: one of the most important issues in designing networks which have to remain functional when component failures occur

- Survivable network design: removal of specific number of vertices and edges does not disconnect the network
  - a railway network with both some tracks and depots being destroyed
  - a traffic network with congestions on some roads and/or intersections
  - telecommunication networks with failures of connections/stations

- How to design a survivable network such that it is still connected after failures of some network components?

Edge- and vertex-connectivity

- $G = (V, E)$: V, set of vertices; E, set of edges.
- l-edge-connected: a graph which remains connected whenever fewer than l edges are removed
  - edge-connectivity: the largest l for which the graph is l-edge-connected

Edge- and vertex-connectivity

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- l-edge-connected: a graph which remains connected whenever fewer than l edges are removed
  - edge-connectivity: the largest l for which the graph is l-edge-connected
- k-vertex-connected: a graph which remains connected whenever fewer than k vertices are removed
  - vertex-connectivity: the largest k for which the graph is k-vertex-connected

Outline

- Mixed connectivity
- IP formulation
- Valid inequalities
- Cutting plane algorithm
- Numerical experiments
- Conclusion

Mixed connectivity

- (k, l)-connected: A graph G is (k, l)-connected, if removal of p vertices and q edges, where

$$p \leq k-1, q \leq l \quad \text{or} \quad p \leq k, q \leq l-1,$$

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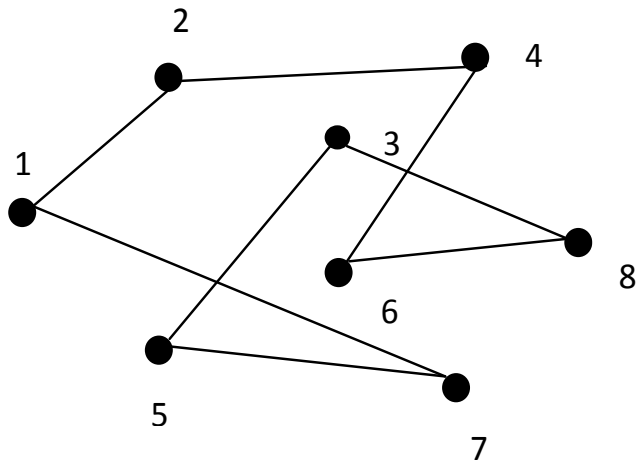
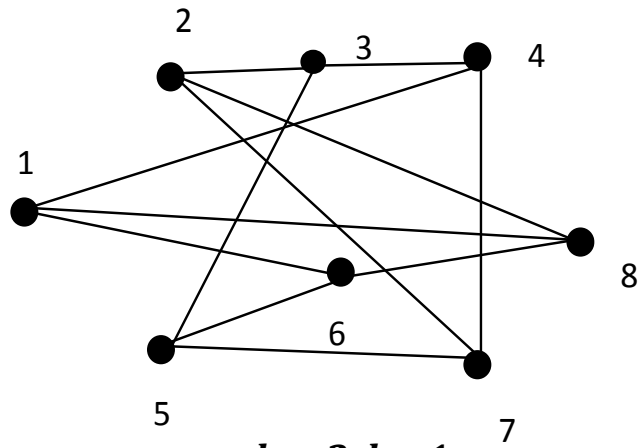
  does not disconnect the graph.
- (k,l)-connectivity: A graph G has (k,l)-connectivity if any pair of vertices are (k,l)-connected with at least one pair not being $(k+1,l)$- nor $(k,l+1)$-connected

  - graph has $(k,0)$-connectivity iff the vertex-connectivity is k
  - graph has $(0,l)$-connectivity iff the edge-connectivity is l

Example



$k = 1, l = 1$



$k = 2, l = 1$

Network design problem

- Problem: Given a graph $G = (V, E)$ with a nonnegative cost c_{ij} associated with each edge $(i, j) \in E$, we want to find minimum-cost (k, l)-connected spanning subgraph.

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- Problem: Given a graph $G = (V, E)$ with a nonnegative cost c_{ij} associated with each edge $(i, j) \in E$, we want to find minimum-cost (k, l)-connected spanning subgraph.
  - *minimum spanning tree*: $k = 0, l = 1$
  - *minimum-cost k-vertex connected subgraph*: $k \geq 2, l = 0$
  - *minimum-cost l-edge connected subgraph*: $k = 0, l \geq 2$

- The minimum-cost (k, l)-connected spanning subgraph problem is NP-hard.

(k, l)-connected graph

Theorem 1

A graph is (k, l)-connected if and only if the resulted graph after removal of any k vertices and $l-1$ edges is connected.

- Lemma 1: If a graph remains connected after removal of any k vertices and $l-1$ edges, then it also remains connected after removal of any $k-1$ vertices and any l edges (Erves and Zeronnik, 2010)
- Proof of Theorem 1:
  - $p \leq k-1, q \leq l$ or $p \leq k, q \leq l-1$
  - $p = k-1, q = l$ or $p = k, q = l-1$
  - according to Lemma, $p = k, q = l-1$

Conditions for (k, l)-connected graph

Theorem 2

A graph $G = (V, E)$ is (k, l)-connected $(k, l \geq 1)$ iff

(i) *G is $(k+1)$-vertex-connected and*

(ii) *G is $(k+l)$-edge-connected.*

Lemma 2: (Menger's Theorem, 1927)

- G is k-vertex-connected iff every pair of vertices is connected by at least k vertex-disjoint paths
- G is l-edge-connected iff every two vertices are connected by at least l edge-disjoint paths

Proof of Theorem 2

- Lemma 3: If a graph G is k-vertex-connected, it is k-edge-connected.
- Lemma 4: If a graph G is (k, l)-connected, it is $(0, k+l)$-connected.
  - (k, l)-connected $\Rightarrow$ resulted graph is still connected after removal of $k-1$ vertices and l edges $\Rightarrow^{Thm\ 1}$ $(k-1, l+1)$-connected $\Rightarrow$ $(k-2, l+2)$-connected $\Rightarrow \cdots\cdots \Rightarrow$ $(0, k+l)$-connected

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- Proof of Theorem 2: "$\Rightarrow$"
  - G is (k, l)-connected so it is still connected after removal of any k vertices. Thus it is $(k+1)$-vertex-connected and condition (i) is proved.

  - According to Lemma 4, G is $(0, k+l)$-connected. So G is $(k+l)$-edge-connected and condition (ii) is proved.

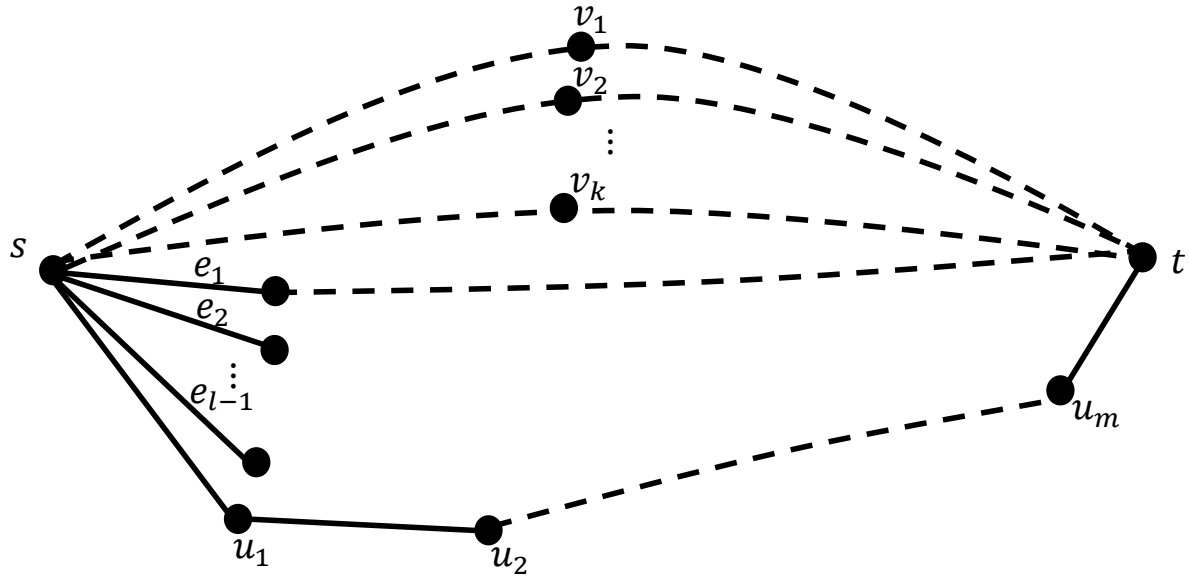
Proof of Theorem 2 (cont.)

- "$\Leftarrow$"

For any pair $s, t \in V$, we want to show, they would be connected after removal of any k vertices and $(l-1)$ edges. According to condition (i), G is $(k+1)$-vertex connected:

- Case 1: By removal of any k vertices, from condition (i), there exists at least one path P from s to t. If none of the $(l-1)$ removed edges are in the path P, the connectivity between s and t is guaranteed.
- Case 2: By removal of any k vertices $(v_1, \cdots, v_k)$ and $(l-1)$ edges $(e_1, \cdots, e_{l-1})$, assume that $(k+1)$-vertex-disjoint paths between s and t are destroyed.

Proof of Theorem 2 (cont.)



- the degree of s is at least $k + l$
- there exists a vertex u_1, such that $(s, u_1) \in E$
- there exists a vertex u_2 such that the edge (u_1, u_2) exists in the resulted graph after removal of $v_1, \cdots, v_k, e_1, \cdots, e_{l-1}$
- continuing this process until we find (u_m, t)
- $(s, u_1), (u_1, u_2), \cdots, (u_m, t)$ forms a path between s and t

Corollary of Theorem 2

- Corollary: *A graph is (k, l)-connected ($k \geq 0, l \geq 1$) if and only if it has $k + l$ edge-disjoint paths between every pair of its vertices, of which $k + 1$ paths are vertex-disjoint.*

- Beineke and Harry (1967) claimed properties for (k, l)- connected subgraphs but Mader (1979) found a gap in their proof
- a conjecture in Enomote and Kaneko (1994), and Kaneko and Ota (2000)

Stronger conditions on (k, l)-connected

Theorem 3

Assume $k \geq 1, l \geq 1$ and $k \leq \frac{n}{2} - 1$, and the vertex set V of G is divided into $V_1 = \{1, \cdots, n-k-1\}$ and $V_2 = \{n-k, \cdots, n\}$. A graph G is (k, l)-connected if and only if it has $k+l$ edge-disjoint paths between every vertex $i \in V_1$ and every vertex $j \in V_2$, of which $k+1$ paths are vertex-disjoint.

- Proof. $\Leftarrow$
  - Case 1. $i \in V_1, j \in V_2$: $(k+l)$ edge-disjoint paths between i and j, of which $(k+1)$ are vertex-disjoint
  - Case 2. $i, j \in V_1$: There exists at least one vertex j' in V_2. i, j are both connected to j'.
  - Case 3. $i, j \in V_2$: If all removed k vertices are in V_2 and removed edges are any ones in G, there exists one vertex $i' \in V$ such that both i, j are connected to it.

- Number of pairs considering in Theorem 2 (or its corollary) is $\frac{n(n-1)}{2}$, while the number of pairs in Theorem 3 is $(n-k-1)(k+1)$.

IP formulation for (k, l)-connected subgraph

$$\min \sum_{(i,j)\in E} c_{ij} x_{ij}$$

$$s.t. \sum_j y_{ij}^{(st)l'} - \sum_j y_{ji}^{(st)l'} = \begin{cases} 1, & i = s \\ -1, & i = t \\ 0, & i \in V \setminus \{s,t\} \end{cases}$$

$$\forall s \in V_1, t \in V_2, l' = 1, \cdots, k+l$$

$$\sum_{l'=1}^{k+l} (y_{ij}^{(st)l'} + y_{ji}^{(st)l'}) \leq x_{ij}, \ \forall s \in V_1, t \in V_2, (i,j) \in E$$

$$\sum_{l'=1}^{k+1} \sum_j y_{ij}^{(st)l'} \leq 1, \ \forall s \in V_1, t \in V_2, \forall i \in V \setminus \{s,t\}$$

$$y_{ij}^{(st)l'} \in \{0,1\}, \ \forall s \in V_1, t \in V_2, (i,j) \in E \cup E', l' = 1, \cdots, k+l$$

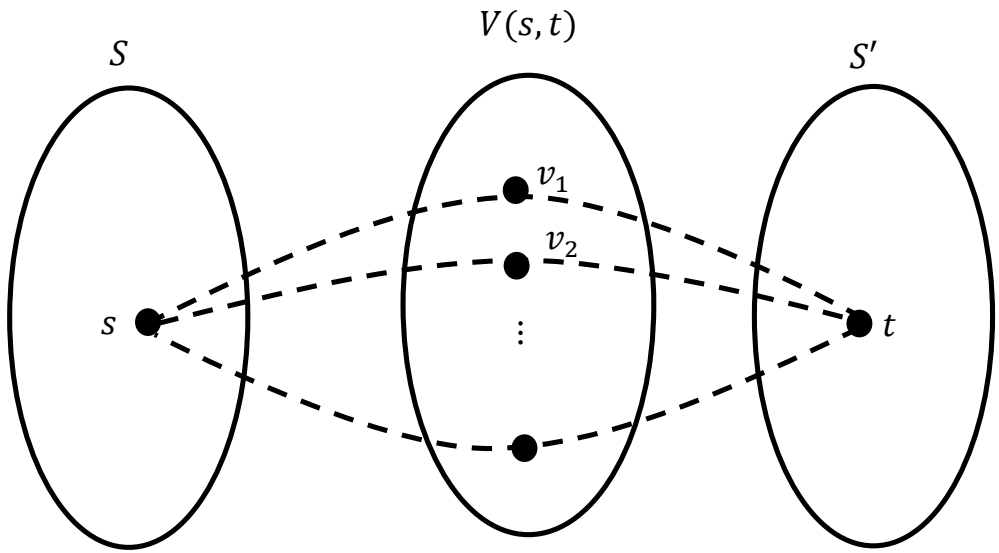
$$x_{ij} \in \{0,1\}, \ \forall (i,j) \in E$$

- $(k+l)$ paths for any pair of vertices (s,t)
- $(k+l)$ paths between s and t are edge-disjoint
- $(k+1)$ paths between s and t are vertex-disjoint

Valid inequalities

- Degree-cut:
  - $\sum_{j:j\in V} x_{ij} \geq (k+l),\ \forall i \in V$

- Vertex cutset
  - $V(s,t)$ be the minimum (s,t)-vertex-cutset
  - $|V(s,t)| < (k+1)$
  - $S \subset V \setminus V(s,t)$ and $S' \subset V \setminus (S \cup V(s,t))$
    - $\sum_{i\in S, j\in S'} x_{ij} \geq (k+1) - |V(s,t)|$



Cutting plane algorithm

Algorithm *Cutting Plane Algorithm for (k,l)-subgraph problem* (**Cutset**)

Input: A graph $G=(V,E)$ with cost matrix $(c_{ij})_{n\times n}$, and two integers k,l

Output: minimum-cost (k,l)-subgraph of G

1: $t \leftarrow 0$

2: solve the problem consisting of objective functions and binary variables x and y

3: obtain optimal solutions x^t

4: for each pair of vertices check the maximum edge-disjoint paths and vertex-disjoint paths by Ford-Folkerson algorithm

5: **if** maximum edge-disjoint paths $< k+l$ **for** a division $(S, V\setminus S)$ $(S\subset V, S\neq\emptyset)$

6: add cut $\sum_{i\in S, j_{V'\setminus S}} x_{ij}^t \geq k+l$ to the problem in Step 2

7: **if** maximum vertex-disjoint paths $< k+1$

8: determine $V(s,t)$ and for a division $(S\setminus V(s,t), V\setminus(S\bigcup V(s,t))$ $(S\subset V, S\neq\emptyset)$

9: add cut $\sum_{i\in S, j\in S'} x_{ij} \geq (k+1) - |V(s,t)|$ to the problem in Step 2

10: **goto** Step 2, $t \leftarrow t+1$

11: **if** no cut added in Steps 4-9, **exit**; x^t is optimal.

Numerical experiments

- All MIP formulations are implemented in C++ using CPLEX 12.3 via IBM's Concert Technology library, version 2.9.
- All experiments were performed on a Linux workstation with 4 Intel(R) Core(TM)2 CPU 2.40GHz processors and 8GB RAM.

Numerical experiments

$\|V\|$	$\|E\|$	k	l	Cutting plane algorithm	path-formulation
10	45	1	3	0.02	0.38
			5	0.02	0.46
			7	0.01	0.83
		3	3	0.02	0.7
			5	0.01	1.32
		5	3	0.01	1.5
15	105	1	3	0.02	0.8
			5		1.67
			11	0.02	2.38
			13	0	2.63
		3	3	0.05	2.98
			9	0.01	3.65
			11	0.01	4.65
		7	5	0.01	3.01
			7	0	4.17
		11	3	0	5.25
20	190	1	3	0.15	18.02
		3	3	0.09	30.18
			5	0.04	118.9
		5	3	0.11	120.57
			9	0.02	97.82
		9	3	0.02	120.99
			9	0.01	410.84
		11	3	0.01	86.21
			7	0.02	408
		15	3	0.01	410.85
30	435	1	3	0.96	28.6
		3	3	0.52	112.66
			5	0.49	416.3
			9	0.13	658.2
		5	3	0.92	365.56
			5	0.29	471.05
			9	0.09	-
			13	0.03	-
		9	3	0.17	-
			5	0.09	-
			9	0.02	-
			13	0.02	-
		11	3	0.07	-
			9	0.02	-
			13	0.02	-
		15	3	0.04	-
			9	0.02	-
			13	0.03	-
		17	3	0.04	-
			9	0.03	-
			11	0.03	-
		21	3	0	-
			5	0.02	-
			7	0.02	-
		23	3	0.03	-
40	780	1	3	1.27	-
		3	3	9.59	-
			5	0.08	-
			9	0.03	-
		5	3	7.28	-
			5	1.1	-
			9	0.09	-
			13	1.43	-
		9	3	1.21	-
			5	0.62	-
			9	1.55	-
			13	0.04	-
		11	3	0.78	-
			9	0.03	-
			13	0.02	-
		15	3	0.5	-
			9	0.03	-
			13	0.03	-
		17	3	0.03	-
			9	0.04	-
			11	0.02	-
		21	3	0.04	-
			5	0.04	-
			7	0.02	-
		23	3	0.05	-
			5	0.08	-
50	1225	1	3	1.53	-
		3	3	9.53	-
			9	0.18	-
		11	3	3.72	-
			9	0.67	-
		15	3	1.73	-
			5	0.88	-
		17	3	2.15	-
			5	OM	-

Note: "OM" means out of memory.

Conclusion

- proposed the network design problem with mixed connectivity requirements, as the minimum-cost (k, l)-connected subgraph problem
- proposed and proved conditions by edge/vertex-disjoint paths for this problem
- developed stronger conditions, and formulated as integer programs
- designed a cutting plane algorithm

Thank You!

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