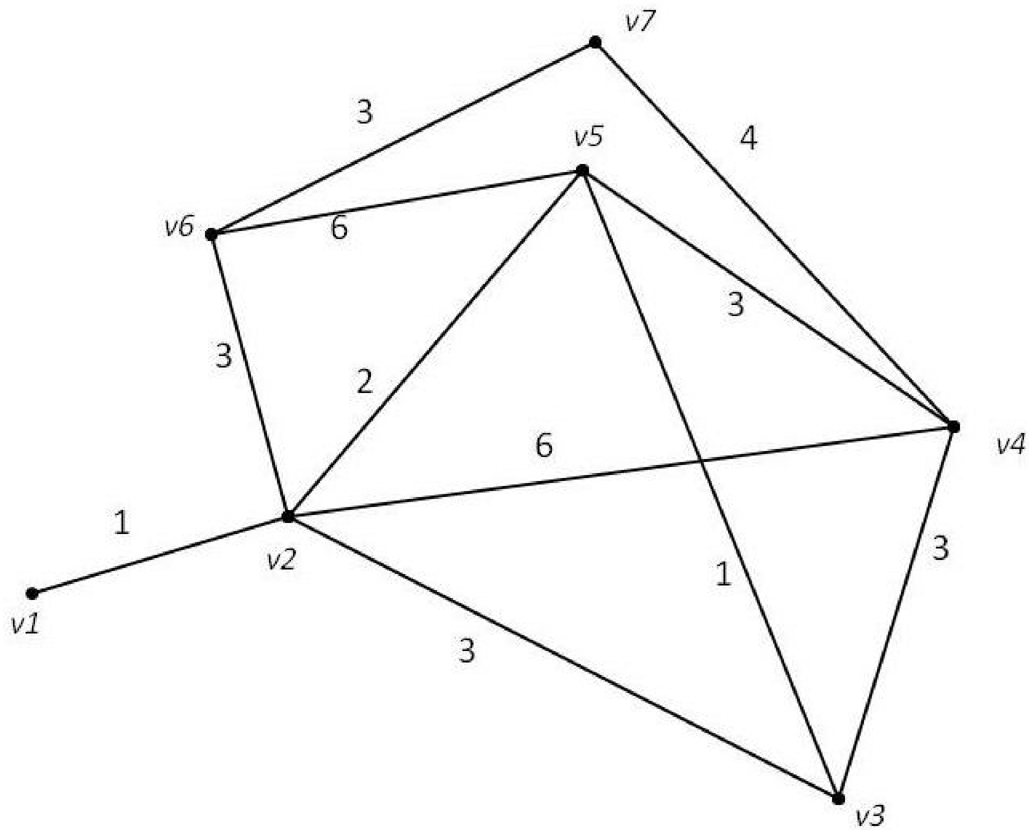
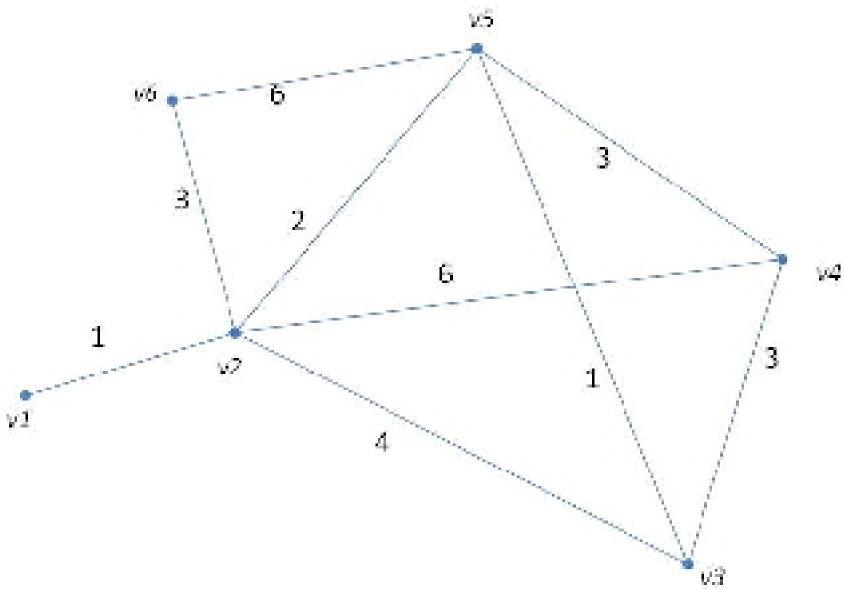
Test 2 Practice

November 26, 2014

1. Find a minimal spanning tree of the following graph.



2. Use Dijkstra's Algorithm to compute the distance between $v1$ and $v3$ in the following network:



YOU DO NOT NEED TO COMPUTE <u>ALL</u> DISTANCES FROM $v1$! Use the following table (you may not need all of it):

$v1$	$v2$	$v3$	$v4$	$v5$	$v6$

3.

a. Show that any tree with at least two vertices must have a vertex of degree 1. Hint: consider the sum of all of the degrees.

b. Suppose I have a tree T_p with vertices $v_1, \dots, v_p$. Further, suppose that v_1 is a vertex of degree 1, and it is connected by an edge to vertex v_2. Explain the difference between the Laplacian matrix for T_p and the Laplacian matrix for T_{p-1}, which is the tree $T_{p-1} = T_p - v_1$.

c. Show directly that the tree T_2 with two vertices has $t(T_2) = 1$ (Do not use the result that $t(G)$ is the number of spanning tree is the number of spanning trees of G).

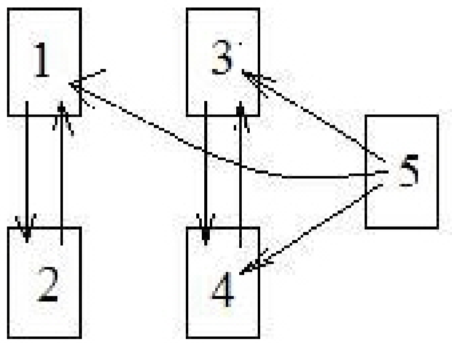
d. (Extra Credit) Use the previous statements to show inductively that any tree T has $t(T) = 1$. (Do not use the result that $t(G)$ is the number of spanning tree is the number of spanning trees of G). Hint: you may use the fact that

$$\det \begin{pmatrix} a_{11}+r & a_{21} & a_{32} & \cdots & a_{n1} \\ a_{12} & a_{22} & \ddots & & a_{n2} \\ a_{13} & \ddots & \ddots & & \vdots \\ \vdots & & & & \vdots \\ a_{1n} & a_{2n} & \cdots & \cdots & a_{nn} \end{pmatrix} = \det A + r \det \hat{A}_{11}$$

where $A = (a_{ij})$ and $\hat{A}_{11}$ is the matrix A with the 1st row and 1st column removed.

4.

Consider the web shown below.



a. Give the link matrix A for the web.

b. Show that the link matrix A has $\dim V_1(A) > 1$.

c. Why does this not contradict the fact that the link matrix A of a strongly connected web has $\dim V_1(A) = 1$? Justify your answer.

5.

Consider the web represented by the following link matrix:

$$A = \begin{pmatrix} 0 & 1 & \frac{1}{5} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{5} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{1}{2} & 0 & 1 \\ \frac{1}{3} & 0 & \frac{1}{5} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

a. Draw a picture of the web

b. The first few powers of A are

$$A^2 = \begin{pmatrix} \frac{2}{5} & 0 & \frac{1}{5} & 0 & 0 & 0 \\ \frac{1}{15} & \frac{1}{3} & \frac{1}{15} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{15} & 0 & 0 & 0 \\ \frac{1}{15} & 0 & \frac{1}{5} & \frac{1}{2} & 0 & 1 \\ \frac{2}{5} & 0 & \frac{3}{10} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{15} & \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} \frac{1}{15} & \frac{2}{5} & \frac{2}{25} & 0 & 0 & 0 \\ \frac{2}{15} & \frac{1}{15} & \frac{2}{25} & 0 & 0 & 0 \\ \frac{2}{15} & 0 & \frac{1}{15} & 0 & 0 & 0 \\ \frac{2}{5} & \frac{1}{15} & \frac{47}{150} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{10} & \frac{2}{5} & \frac{7}{25} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{15} & \frac{9}{50} & \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}, \quad A^4 = \begin{pmatrix} \frac{4}{25} & \frac{1}{15} & \frac{7}{75} & 0 & 0 & 0 \\ \frac{11}{225} & \frac{2}{15} & \frac{1}{25} & 0 & 0 & 0 \\ \frac{1}{45} & \frac{2}{15} & \frac{2}{75} & 0 & 0 & 0 \\ \frac{19}{150} & \frac{2}{5} & \frac{22}{75} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{59}{150} & \frac{1}{10} & \frac{7}{20} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ \frac{56}{225} & \frac{1}{6} & \frac{59}{300} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}$$

Suppose I walk through the web step by step so that if I am at a page, the next step I choose one of the outlinks with equal probability. What is the probability that, starting at page 2, I end up at page 3 in 4 steps?

c. Using this interpretation and your picture of the web, explain why the top right corner entry of A^n is zero for any power n. (Hint: Consider the subweb of pages 4, 5 and 6 only.)

6.

a. Consider a 4096×4096 grid (note that $4096 = 2^{12}$). Suppose every vertex has two additional edges, each attached to a random vertex (so for each of the two additional edges from u, the probability of it being edge (u, v) is $P_{(u,v)} = \frac{1}{4096^2} = \frac{1}{2^{24}}$). Explain why there are probably paths between any two vertices of length less than or equal to 24.

b. Suppose we use the myopic search algorithm (decentralized search) on the grid considered in part a. Recall the following:

- The message is in phase 10 if it is a distance less than $2^{11} = 2048$ and more than $2^{10} = 1024$ from the target.
- The probability P_{10} that the message at vertex u in phase 10 jumps to B_{10} (the ball around the target of radius $2^{10} = 1024$) in one step is

$$P_{10} = \sum_{v \in B_{10}} 2P_{(u,v)}.$$

- There are at least $\frac{1}{2}2^{2(10)} = 2^{19}$ vertices in B_{10}.

Show that

$$P_{10} \geq 2\frac{2^{19}}{2^{24}} = \frac{1}{2^4} = \frac{1}{16}.$$