COHOMOLOGY

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1. Introduction

Cohomology is the homology theory gotten by the dual chain complex to homology:

$$C_n^* = \operatorname{Hom}(C_n, \mathbb{Z}),$$

which has coboundary maps $\delta_n = \partial_{n+1}^* : C_n^* \to C_{n+1}^*$ (notice that the index goes up instead of down). The main advantage of cohomology over homology is that it is a ring, i.e., it has a multiplication, whereas homology does not (it has a coproduct making it a co-ring, instead of a product making it a ring). The product is a bit difficult to describe, and we will not describe it for singular cohomology.

Instead, we will look at cohomology of the complex of differential forms, which have a natural product, the wedge product.

2. De Rham cohomology groups

Definition 1. *Recall the space of k-forms $\mathcal{A}^k(M)$, together with the differential maps $d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$. The set of closed k-forms $\mathcal{Z}^k(M)$ are forms $\omega \in \mathcal{A}^k(M)$ such that $d\omega = 0$. The set of exact k-forms $\mathcal{B}^k(M)$ are the forms $\omega \in d\mathcal{A}^{k-1}(M) \subseteq \mathcal{A}^k(M)$. The de Rham cohomology groups $H^k_{dR}(M)$ are defined as*

$$H^k_{dR}(M) = \mathcal{Z}^k(M) / \mathcal{B}^k(M).$$

Remark 1. *The k-forms form a cochain complex given by the differential maps $d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$. Notice that in a cochain complex, the differential increases the index, while in a chain complex the index is decreased. Cochain complexes give cohomology and chain complexes give homology.*

Remark 2. *Recall that $\mathcal{A}^k(M)$ is a vector space over $\mathbb{R}$. The singular chains $C_k(M)$ are free abelian groups, and hence modules over $\mathbb{Z}$. The groups $\mathcal{A}^k(M)$ have more similarity to $C_k(M) \otimes \mathbb{R}$, which is a vector space.*

It will be important to recall the following fact from last semester (the proof is a direct calculation, and you can refer to Lemma 12.16 in Lee):

Proposition 2. *If $G : M \to N$ is a smooth map, then the pullback map $G^* : \mathcal{A}^k(N) \to \mathcal{A}^k(M)$ commutes with d, i.e., $dG^* = G^* d$.*

We now look at the same sorts of "functorial" properties of the cohomology. Notice that induced maps on cohomology naturally turn compositions around, as opposed to induced maps on homology.

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Proposition 3. *For any smooth map $G : M \to N$, the pullback $G^* : \mathcal{A}^k(N) \to \mathcal{A}^k(M)$ carries closed forms to closed forms and exact forms to exact forms. Thus it induces a homomorphism on cohomology. It has the following properties:*

(1) *If $F : N \to P$ is another smooth map, then*
$$(F \circ G)^* = G^* \circ F^*.$$
(2) *If Id_M denotes the identity map on M, then $Id_M^* : H^p_{dR}(M) \to H^p_{dR}(M)$ is the identity map.*

Proof. Recall that pullbacks commute with the differential, i.e., $G^* d\omega = dG^*\omega$, which we recall follows from the calculation:
$$\begin{aligned} G^* d\left(f\ dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) &= G^*\left(df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) \\ &= d\left(f \circ G\right) \wedge d\left(x^{i_1} \circ G\right) \wedge \cdots \wedge d\left(x^{i_k} \circ G\right) \\ &= dG^*\left(f\ dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right). \end{aligned}$$

[REMIND YOURSELF OF HOW TO COMPUTE PULLBACKS AND DIFFERENTIALS!] Thus we get that if ω is closed, i.e., $d\omega = 0$, then
$$dG^*\omega = G^* d\omega = 0$$
and if $\omega = d\eta$ (ω is exact), then
$$G^*\omega = G^* d\eta = dG^*\eta$$
and so $G^*\omega$ is exact. It follows that it induces a homomorphism on cohomology.

The other two follow from their properties on forms. □

Corollary 4. *Diffeomorphic manifolds have isomorphic de Rham cohomology groups.*

Proposition 5. *There is a product $H^k_{dR}(M) \times H^\ell_{dR}(M) \to H^{k+\ell}_{dR}(M)$ given by*
$$[\omega][\tau] = [\omega \wedge \tau]$$
that gives $\bigoplus_k H^k_{dR}(M)$ a (graded) ring structure.

Proof. We will just show that the product is well-defined. Suppose $d\omega = 0$ and $d\tau = 0$. Consider
$$\begin{aligned} [\omega + d\omega'][\tau + d\tau'] &= [(\omega + d\omega') \wedge (\tau + d\tau')] \\ &= [\omega \wedge \tau + \omega \wedge d\tau' + d\omega' \wedge \tau + d\omega' \wedge d\tau'] \\ &= [\omega \wedge \tau] \end{aligned}$$
since
$$\begin{aligned} d\left((-1)^k \omega \wedge \tau' + \omega' \wedge \tau + \omega' \wedge d\tau\right) &= (-1)^k d\omega \wedge \tau' + \omega \wedge d\tau' + d\omega' \wedge \tau + (-1)^{k-1} \omega' \wedge d\tau + d\omega' \wedge d\tau' \\ &= \omega \wedge d\tau' + d\omega' \wedge \tau + d\omega' \wedge d\tau' \end{aligned}$$
since $d\omega = 0$ and $d\tau = 0$. □

3. Homotopy invariance

Proposition 6. *Let $F, G : M \to N$ be (smoothly) homotopic smooth maps. For every p, the induced cohomology maps $F^*, G^* : H^p_{dR}(N) \to H^p_{dR}(M)$ are equal.*

Proof. This proof is similar to the proof of homotopy invariance of maps between homology groups. We need to show that the homotopy induces a cochain homotopy equivalence, which is a map $h : \mathcal{A}^p(N) \to \mathcal{A}^{p-1}(M)$ such that $dh\omega + hd\omega = G^*\omega - F^*\omega$. (Check that this implies the induces maps on homology are the same. The argument is analogous to chain homotopy equivalence.) Let $H : M \times I \to N$ be a smooth homotopy between F and G. Let $\omega \in \mathcal{A}^p(N)$. We can pull back $H^*\omega \in \mathcal{A}^p(M \times I)$ and let

$$h\omega = \int_0^1 \frac{\partial}{\partial t} \lrcorner H^*\omega \; dt$$

where $\frac{\partial}{\partial t}$ is the generator for TI. This goes the proper space. Now suppose that $H^*\omega = f dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}}$ and we check:

$$\begin{aligned}
dh\omega &= d\left(\int_0^1 \frac{\partial}{\partial t} \lrcorner H^*\omega \; dt\right) \\
&= d\left(\left(\int_0^1 f(x,t)\, dt\right) dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}}\right) \\
&= \left(\int_0^1 \frac{\partial f}{\partial x^j} dt\right) dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}}.
\end{aligned}$$

Now note that

$$\begin{aligned}
H^*d\omega &= dH^*\omega \\
&= d\left(f dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}}\right) \\
&= \frac{\partial f}{\partial x^j} dx^j \wedge dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}}
\end{aligned}$$

so

$$\begin{aligned}
hd\omega &= \int_0^1 \frac{\partial}{\partial t} \lrcorner \frac{\partial f}{\partial x^j} dx^j \wedge dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}} \; dt \\
&= -\left(\int_0^1 \frac{\partial f_j}{\partial x^j} dt\right) dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}}
\end{aligned}$$

and so in this case $dh\omega + hd\omega = 0$. In general, we may have a term in $H^*\omega$ that has the form $f dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. For such a term, certainly $h\omega = 0$, but

$$\begin{aligned}
hd\omega &= \int_0^1 \frac{\partial}{\partial t} \lrcorner H^*d\omega \; dt \\
&= \int_0^1 \frac{\partial}{\partial t} \lrcorner dH^*\omega \; dt \\
&= \left(\int_0^1 \frac{\partial f}{\partial t} \; dt\right) dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\
&= [f(x,1) - f(x,0)]\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}.
\end{aligned}$$

In general, if $\omega = g\ dy^{j_1} \wedge \cdots \wedge dy^{j_p}$, then

$$\begin{aligned} H^*\omega &= (g \circ H)\ \ d\left(y^{j_1} \circ H\right) \wedge \cdots \wedge d\left(y^{j_p} \circ H\right) \\ &= (g \circ H) \frac{\partial H^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial H^{j_p}}{\partial x^{i_p}} dx^{i_1} \wedge \cdots \wedge dx^{i_p} + dt \wedge \eta \end{aligned}$$

for some form η. Using the previous computations, we see that

$$\begin{aligned} dh\omega + hd\omega &= \left[(g \circ H) \frac{\partial H^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial H^{j_p}}{\partial x^{i_p}}\right]_0^1 dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ &= F^*\omega - G^*\omega. \end{aligned}$$

□

Theorem 7. *If M, N are smoothly homotopy equivalent smooth manifolds, then $H^p_{dR}(M) \cong H^p_{dR}(N)$ for each p. The isomorphism is induced by any cochain homotopy equivalence.*

4. Mayer-Vietoris

Theorem 8 (Mayer-Vietoris sequence)**.** *Let M be a smooth manifold and let U, V be open subsets of M whose union is M. For each p, there is a linear map $\delta : H^p_{dR}(U \cap V) \to H^{p+1}_{dR}(M)$ such that the following sequence is exact:*

$$\cdots \longrightarrow H^{p-1}_{dR}(U \cap V) \xrightarrow{\delta} H^p_{dR}(M) \xrightarrow{k^* \oplus \ell^*} H^p_{dR}(U) \oplus H^p_{dR}(V) \xrightarrow{i^* - j^*} H^p_{dR}(U \cap V) \xrightarrow{\delta} H^{p+1}_{dR}(M)$$

where $i : U \cap V \to U$, $j : U \cap V \to V$, $k : U \to M$, $\ell : V \to M$ are inclusion maps.

Proof. The long exact sequence is derived from a short exact sequence of cochain complexes:

$$0 \to \mathcal{A}^p(M) \xrightarrow{k^* \oplus \ell^*} \mathcal{A}^p(U) \oplus \mathcal{A}^p(V) \xrightarrow{i^* - j^*} \mathcal{A}^p(U \cap V) \to 0$$

Once it is proven that this is a short exact sequence of cochain complexes, a zigzag lemma/diagram chase produces the long exact sequence, similar to the construction for chain complexes. If $\omega \in \mathcal{A}^p(M)$, $k^*\omega$ and $\ell^*\omega$ are just the restrictions of the forms to U and V respectively, and hence if both are zero, then ω is zero on M, so the first map is injective. Now suppose $(\alpha, \beta) \in \ker(i^* - j^*)$. Then $i^*(\alpha) = j^*(\beta)$, and so the restrictions of the two forms α and β to $U \cap V$ must be the same, so the two forms can be extended to a smooth form on $U \cup V = M$, and since it is clear that $(i^* - j^*) \circ k^* \oplus \ell^* = 0$, the next part is exact.

Finally, we need to show that for any $\omega \in \mathcal{A}^p(U \cap V)$, there are forms $\alpha \in \mathcal{A}^p(U)$ and $\beta \in \mathcal{A}^p(V)$ such that $i^*\alpha - j^*\beta = \omega$. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to $\{U, V\}$. We can now define

$$\alpha = \begin{cases} \rho_V \omega \text{ on } U \cap V \\ 0 \text{ on } U \setminus \operatorname{supp} \rho_V \end{cases}$$
$$\beta = \begin{cases} -\rho_U \omega \text{ on } U \cap V \\ 0 \text{ on } V \setminus \operatorname{supp} \rho_U \end{cases}$$

which are smooth forms on U and V. We now see that $i^*\alpha - j^*\beta = (\rho_U + \rho_V)\omega = \omega$. □

Note that we can again get an understanding of the connecting homomorphism. If $\delta[\omega] = [\sigma]$, then there exist $\alpha \in \mathcal{A}^p(U)$ and $\beta \in \mathcal{A}^p(V)$ such that $i^*\alpha - j^*\beta = \omega$ and $k^*\sigma = d\alpha$ and $\ell^*\sigma = d\beta$. If we define α and β as in the proof (using the partitions of unity), we simply need to find $\sigma \in \mathcal{A}^p(M)$. Note that since $d\omega = 0$ on $U \cap V$, we have that

$$d\alpha|_{U\cap V} = d(\omega + \beta_{U\cap V}) = d\beta|_{U\cap V}.$$

It follows that the support of $d\alpha$ is in $U \cap V$, and so it can be extended to a smooth form on M, and we call that form σ. Note $[d\alpha] = [d\beta]$.

5. Computations

Proposition 9. *Let $M = \coprod_i M_i$ be a disjoint union of smooth manifolds, the inclusions $\iota_j : M_j \to M$ induce an isomorphism from $H^p_{dR}(M)$ to the direct product $\prod_j H^p_{dR}(M_j)$.*

Proof. These are already an isomorphism on forms, since if $\omega \in \mathcal{A}^p(M)$ and $\iota_j^*\omega = 0$ for all $\mathcal{A}^p(M_j)$, then $\omega = 0$. Also, for any element in the direct product, the product gives a form on M. □

Proposition 10. *If M is a connected smooth manifolds, then $H^0_{dR}(M)$ is equal the the space of constant functions, and hence one-dimensional.*

Proof. A closed 0-form is a function f such that $df = 0$. On a connected smooth manifold, this means that f is a constant. □

Proposition 11. *If M is a zero-dimensional manifold, the dimension of $H^0_{dR}(M)$ is equal to the cardinality of M, and all other de Rham cohomology groups are zero.*

Proof. Since a zero-dimensional manifold is a disjoint union of connected zero-manifolds (i.e., points), the result follows. □

Proposition 12 (Poincare Lemma)**.** *Let U be a star-shaped open subset of $\mathbb{R}^n$. Then $H^p_{dR}(U) = 0$ for $p \geq 1$.*

Proof. Star shaped domains are contractible to a point, and so $H^p_{dR}(U) \cong H^p_{dR}(\{pt\}) = 0$ for $p \geq 1$. □

Proposition 13. *For all $p \geq 1$, $H^p_{dR}(\mathbb{R}^n) = 0$.*

Proof. $\mathbb{R}^n$ is star-shaped. □

Proposition 14. *Let M be a smooth manifold, and let ω be a closed p-form on M, with $p \geq 1$. The for every $q \in M$, there is a neighborhood U of q on which ω is exact.*

Proof. Every point has a neighborhood U diffeomorphic to a star-shaped domain in $\mathbb{R}^n$. We thus have that $H^p_{dR}(U) = 0$, so given ω such that $d\omega = 0$, we also have that the restriction of ω to U satisfies $d\omega = 0$ and since $[\omega] = 0$, we must have a form $\eta \in \mathcal{A}^{p-1}(U)$ such that $\omega = d\eta$. □

One can also compute the cohomology of spheres using Mayer-Vietoris in much the same way as we did for the homology of spheres. Note that we have the form $d\theta$ (which is not closed since we are considering the extension of the differential of the angle function), that we showed is closed but not exact last semester. One needs to show that this generates the homology of S^1, and then use Mayer-Vietoris to propagate the generator to all S^n. Let's consider what happens for S^1, where U and V are extensions of the top and bottom of S^1, so the intersection deformation retracts to S^0 =two points. Note that $H^0_{dR}\left(S^0\right)$ is generated by constant functions $(0,1)$ and $(1,0)$, where the comma refers to the value of the functions on each component. Now, if we consider the function $f=(1,1)$, clearly this extends to the function $f=1$ on both U and V, but since this is constant, we have $df=0$. However, if we consider, say $(1,0)$, then this can be extended to the function $f(\theta)=1-\theta$ where θ goes from 0 to 1. If we check carefully, we see that the image of this under δ is $d\theta$.

Using the usual induction argument, we get the following.

Proposition 15. *The de Rham cohomology of spheres is*

$$H^p_{dR}(S^n) \cong \begin{cases} \mathbb{R} & \text{if } p=0,n \\ 0 & \text{otherwise.} \end{cases}$$

Recall that we know how to integrate n-forms over the manifold, and Stokes' theorem says that $\int_M d\omega = 0$ if M is a closed manifold. Thus, we get that the integration map

$$I([\omega]) = \int_M \omega$$

is well-defined on de Rham cohomology. Note the following:

Proposition 16. *An n-form ω on S^n is exact if and only if $\int_{S^n}\omega = 0$.*

Proof. Clearly, if $\omega = d\eta$, then by Stokes' Theorem, we have that $\int_{S^n}\omega = 0$.

Since every n-form is closed, we if ω is not exact, then it generates a nontrivial de Rham cohomology class. By the calculation of the cohomology of spheres, there must be an orientation form Ω such that $\omega = c\Omega + d\eta$. The integral of this is c, and so if $c \neq 0$, we have that the integral is nonzero. □

Proposition 17. *For any balls $B \subset B'$ in $\mathbb{R}^n$, an $(n-1)$-form ω on $\mathbb{R}^n \setminus \bar{B}$ is exact if and only if $\int_{\partial B'}\omega = 0$.*

Proof. We know that $\mathbb{R}^n \setminus \bar{B}$ is homotopic to S^{n-1}, and so the de Rham cohomology of $\mathbb{R}^n \setminus \bar{B}$ is the same as that of S^{n-1}. Moreover, the inclusion map $\iota : \partial B' \to \mathbb{R}^n \setminus \bar{B}$ generates an isomorphism of the de Rham cohomology groups, and so ω is exact if and only if $\iota^*\omega$ is exact if and only if $\int_{S^{n-1}}\iota^*\omega = 0$ by the previous proposition. □

Theorem 18. *For any closed (compact), connected, oriented, smooth n-manifold M, the integration map $I : H^n_{dR}(M) \to \mathbb{R}$ is an isomorphism. Thus $H^n_{dR}(M) \cong \mathbb{R}$ and is spanned by any smooth orientation form.*

Proof. We already know that an orientation form gives a nonzero class in cohomology, since $d\Omega = 0$ trivially and $\int_M \Omega \neq 0$ shows that $\Omega \neq d\omega$ and so $[\Omega] \neq 0$. Thus the integration map is surjective, and we need only show it is injective. Another way to express this is to say that we need to show that $\int_M \omega = 0$ implies that ω is exact. Note that we have already shown this in the case of one-dimensional manifolds (conservative if and only if exact). The zero dimensional case is clear, since $H^0_{dR}(M)$ is generated by functions that are constant on each connected component.

To show injectivity, we need to show that for any n-form ω, if $\int_M \omega = 0$, then ω is exact. We will do this by considering finite covers by coordinate balls. Unfortunately, coordinate balls are not compact, so to integrate, we must consider compactly supported forms. In the end, it is okay since all forms on a compact manifold are compactly supported. Let $\{U_1, \ldots, U_k\}$ be a finite cover of M by coordinate balls. We will use Lemma 19 below and build up showing that compactly supported forms on $M_j = \bigcup_{i=1}^{j} U_i$ that integrate to zero are exact. Lemma 19 gives the base case. Now suppose it is true for j and consider a compactly supported form ω on $M_{j+1} = \bigcup_{i=1}^{j+1} U_i$ such that $\int_M \omega = 0$. Let $\{\phi, \psi\}$ be a partition of unity subordinate to the covering $\{M_j, U_{j+1}\}$ of M_{j+1}. We will need an auxiliary form to adjust the integral, since $\phi\omega$ and $\psi\omega$ do not each integrate to zero, so let Ω be a n-form compactly supported in $M_j \cap U_j$ such that $\int_M \Omega = 1$. (Note: we can reorder the U_i so that this intersection is always nonempty, and we can find Ω using a bump function and the orientation form.) We can now consider $\phi\omega - c\Omega$ and $\psi\omega + c\Omega$ where $c = \int_{M_j} \phi\omega$. Both are compactly supported in M_j and U_{j+1} respectively and

$$\int_{M_j} (\phi\omega - c\Omega) = 0$$
$$\int_{U_{j+1}} (\psi\omega + c\Omega) = \int_{M_j} ((1-\phi)\,\omega + c\Omega) = 0.$$

Thus there are compactly supported $(n-1)$-forms η_1 and η_2 such that $\phi\omega - c\Omega = d\eta_1$ and $\psi\omega + c\Omega = d\eta_2$ (by the inductive hypothesis and Lemma 19). It follows that $\eta = \eta_1 + \eta_2$ satisfies

$$d\eta = \omega.$$

Since $M_k = M$, the proof is complete. □

Lemma 19. *Let $n \geq 1$ and suppose ω is a compactly supported smooth n-form on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \omega = 0$. Then there exists a compactly supported $(n-1)$-form η on $\mathbb{R}^n$ such that $d\eta = \omega$.*

Remark 3. *We already know that an η exists, but not that one that is compactly supported.*

Proof. If $n = 1$, then $\omega = f dx$ for a smooth, compactly supported function f. We can now write down the antiderivative

$$F(x) = \int_{-\infty}^{x} f(t)\,dt,$$

which satisfies $dF = f dx = \omega$. If f is supported in $[-R, R]$, then clearly F is zero for $x < -R$. Since $f = 0$ for $x > R$ and $\int_{-\infty}^{\infty} \omega = 0$, we must have that F is zero for $x > R$.

Now suppose $n \geq 2$. Let B and B' be open balls in $\mathbb{R}^n$ such that

$$\operatorname{supp} \omega \subseteq B \subseteq \bar{B} \subseteq B'.$$

There exists a smooth $(n-1)$-form η_0 such that $d\eta_0 = \omega$. We have

$$0 = \int_{\mathbb{R}^n} \omega = \int_{\bar{B}'} \omega = \int_{\bar{B}'} d\eta_0 = \int_{\partial B'} \eta_0.$$

It follows from Proposition 17 that η_0 is exact on $\mathbb{R}^n \setminus \bar{B}$, so there is a smooth $(n-2)$-form γ on $\mathbb{R}^n \setminus \bar{B}$ such that $\eta_0 = d\gamma$. Now let ψ be a smooth bump function supported on $\mathbb{R}^n \setminus \bar{B}$ that is one on $\mathbb{R}^n \setminus B'$. Then consider

$$\eta = \eta_0 - d(\psi\gamma)$$

This is a compactly supported $(n-1)$-form such that $d\eta = \omega$. □

6. Smooth singular cohomology

Definition 20. *A smooth p-simplex on M is a smooth map $\sigma : \Delta^p \to M$. (Recall that smooth means there is a smooth extension of the map in a neighborhood of any point.) The smooth chain group $C_p^\infty(M) \subseteq C_p(M)$. The cohomology of this chain group with the boundary operator ∂ is called the smooth singular homology group $H_p^\infty(M)$*

Theorem 21. *For any smooth manifold M, the map $H_p^\infty(M) \to H_p(M)$ induced by inclusion is an isomorphism.*

We will omit the proof. The idea is that each singular chain inducing a homology element needs to be approximated by a smooth singular chain that behaves well under the boundary map.

Since we will be dealing only with cohomology with $\mathbb{R}$ coefficients (not $\mathbb{Z}$), we can define homology groups as the dual.

Definition 22. *The singular cohomology groups $H^p(M;\mathbb{R})$ are defined as the dual space $Hom(H_p(M), \mathbb{R})$. Note that the previous theorem allows us to replace understand this simply as the dual of the homology of smooth singular chains.*

Remark 4. *If we were interested in $\mathbb{Z}$, then we would have to consider the dual of the singular chain complex, and then take the cohomology of that cochain complex. Since we are doing only $\mathbb{R}$ coefficients, we do not need to worry about this extra detail.*

Proposition 23. *There is a Mayer-Vietoris sequence for singular cohomology.*

We will omit the proof, which can be found by simply dualizing the proof for singular homology.

7. de Rham's theorem

Definition 24. *Let $\bar{\mathbb{R}}^n_+ = \left\{ (x^1, \ldots, x^n) : x^i \geq 0 \text{ for all } i \right\}$. Then a smooth manifold with corners is a topological manifold with boundary such that each point has a neighborhood diffeomorphic to an open set in $\bar{\mathbb{R}}^n_+$.*

Remark 5. *Most importantly, the simplex Δ^n is a smooth manifold with corners.*

Theorem 25 (Stokes' Theorem for Manifolds with Corners)**.** *Let M be a smooth, oriented n-manifold with corners, and let ω be a compactly supported $(n-1)$-form on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega$$

Remark 6. *This needs a definition, but the point is that for a smooth manifold with corners, we only integrate over parts of the manifold away from the corners in the boundary (i.e., away from codimension 2 and below), and so the boundary makes sense, as does integration over M and ∂M.*

So now we know how to integrate smooth p-forms over smooth p-simplices (since these are manifolds with corners), and we can extend by linearity to integration over smooth p-chains. I.e., if $c = \sum c_i \sigma_i^p$ and ω is a smooth p-form, then

$$\int_c \omega = \sum c_i \int_{\sigma_i^p} \omega = \sum c_i \int_{\Delta^p} (\sigma_i^p)^* \omega.$$

Theorem 26 (Stokes' Theorem for Chains)**.** *If c is a smooth p-chain in a smooth manifold M and ω is a smooth $(p-1)$-form on M, then*

$$\int_{\partial c}\omega=\int_c d\omega$$

This theorem requires checking that the appropriate boundary corresponds to orientation preserving diffeomorphisms, i.e., if σ^p is a smooth p-simplex and $\omega = f dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ then

$$\begin{aligned}\int_{\sigma^p} d\omega &= \int_{\Delta^p}\sigma^* d\omega\\ &= \int_{\Delta^p} d\sigma^*\omega\\ &= \int_{\partial\Delta^p}\sigma^*\omega.\end{aligned}$$

At this point, $\partial\Delta^p$ means the boundary as a submanifold with the induced Stokes' orientation. The claim is that if we consider the smooth singular boundary map $\partial\Delta^p=\sum_i(-1)^i[v_0,\ldots,\hat{v}_i,\ldots,v_p]$, that this is induced by the Stokes' orientation. That is, we need to understand the associated orientations of $[v_0,\ldots,\hat{v}_i,\ldots,v_p]$. To do this, it is easier to associate the standard simplex Δ^p with the simplex $[0,e_1,\ldots,e_p]$ instead of $[e_0,\ldots,e_p]$, so that we can consider it as a subset of $\mathbb{R}^p$. Then each of the faces is a codimension one submanifold and we need to compute the induced Stokes' orientation. Each face map is an affine map (linear plus translation).

The map taking Δ^{p-1} to $[v_0,\ldots,\hat{v}_i,\ldots,v_p]$ is

$$\left(x^1,\ldots,x^{p-1}\right)\to\left(x^1,\ldots,x^{i-1},0,x^{i+1},\ldots,x^{p-1}\right)$$

if $i\neq 0$ and

$$\left(x^1,\ldots,x^{p-1}\right)\to\left(1-\sum_{j=1}^{p-1}x^j,x^1,\ldots,x^{p-1}\right)$$

if $i=0$.

If we choose the orientation $dx^1\wedge\cdots\wedge dx^n$ on $\Delta^p\subseteq\mathbb{R}^p$,we see that for $i>0$, on $[0,\ldots,\hat{e}_i,\ldots,e_p]$, the vector $-e_i$ is outward pointing, and so the face map (map from Δ^{p-1} with orientation $dx^1\wedge\cdots\wedge dx^{p-1}$ to the face) is orientation reversing if i is odd and orientation preserving if i is even. For $i=0$, the vector e_1 is outward pointing, and so the map is orientation preserving.

It follows that the signs given in the boundary map ∂ correspond to whether the face maps are orientation preserving or reversing, and so the integration is well-defined on chains.

Definition 27. *We define the de Rham map $J:H^p_{dR}(M)\to H^p(M;\mathbb{R})$ as follows:*

$$J([\omega])[\sigma]=\int_\sigma\omega.$$

We need to check that this is well-defined, i.e., if we take $\omega + d\eta$ and $\sigma + \partial\tau$ that we get the same answer, but since ω is closed and σ is a cycle, we have

$$\begin{aligned} J\left(\left[\omega + d\eta\right]\right)\left[\sigma + \partial\tau\right] &= \int_\sigma \omega + \int_\sigma d\eta + \int_{\partial\tau} \omega + \int_{\partial\tau} d\eta \\ &= \int_\sigma \omega. \end{aligned}$$

Lemma 28. *If $F : M \rightarrow N$ is a smooth map, then the following diagram is commutative:*

$$\begin{array}{ccc} H^p_{dR}\left(N\right) & \xrightarrow{F^*} & H^p_{dR}\left(M\right) \\ \downarrow J & & \downarrow J \\ H^p\left(N;\mathbb{R}\right) & \xrightarrow{F^*} & H^p\left(M;\mathbb{R}\right) \end{array}$$

Proof. Suppose $[\omega] \in H^p_{dR}\left(N\right)$, so ω is a closed p-form on N. Then for a simplex σ^p_M on M, we compute

$$\int_{\sigma^p_M} F^*\omega = \int_{\Delta^p} \left(\sigma^p_M\right)^* F^*\omega = \int_{\Delta^p} \left(F \circ \sigma^p_M\right)^* \omega = \int_{F\circ\sigma^p_M} \omega.$$

It follows that

$$F^* J\left(\left[\omega\right]\right) = F^*\left[\sigma^p_N \rightarrow \int_{\sigma^p_N} \omega\right] = \left[\sigma^p_M \rightarrow \int_{F\circ\sigma^p_M} \omega\right]$$

and

$$JF^*\left[\omega\right] = J\left[F^*\omega\right] = \left[\sigma^p_M \rightarrow \int_{\sigma^p_M} F^*\omega\right] = \left[\sigma^p_M \rightarrow \int_{F\circ\sigma^p_M} \omega\right].$$

□

Lemma 29. *The de Rham map is natural with respect to the Mayer-Vietoris sequence, i.e., the following diagram is commutative:*

$$\begin{array}{cccccccccc} \cdots \longrightarrow & H^{p-1}_{dR}\left(U\cap V\right) & \xrightarrow{\delta} & H^p_{dR}\left(M\right) & \xrightarrow{k^*\oplus\ell^*} & H^p_{dR}\left(U\right)\oplus H^p_{dR}\left(V\right) & \xrightarrow{i^*-j^*} & H^p_{dR}\left(U\cap V\right) & \xrightarrow{\delta} & \\ & \downarrow J & & \downarrow J & & \downarrow J & & \downarrow J & & \\ \cdots \longrightarrow & H^{p-1}\left(U\cap V;\mathbb{R}\right) & \xrightarrow{\delta} & H^p\left(M;\mathbb{R}\right) & \xrightarrow{k^*\oplus\ell^*} & H^p\left(U;\mathbb{R}\right)\oplus H^p\left(V;\mathbb{R}\right) & \xrightarrow{i^*-j^*} & H^p\left(U\cap V;\mathbb{R}\right) & \xrightarrow{\delta} & H \end{array}$$

Proof. We just need to check the connecting homomorphism. Let $[\omega] \in H^p_{dR}\left(U\cap V\right)$, so ω is a closed form on $U \cap V$. The connecting homomorphism for de Rham cohomology is constructed as follows. We find smooth forms ω_U on U and ω_V on V such that $\omega = \omega_U|_{U\cap V} - \omega_V|_{U\cap V}$, and then $\delta\left[\omega\right] = \left[d\omega_U\right]$ where $d\omega_U = d\omega_V$ can be extended to all of M (and is supported in $U \cap V$). Now if we take J of $\left[d\omega_U\right]$, we get

$$J\delta\left[\omega\right] = \left[\sigma^{p+1} \rightarrow \int_{\sigma^{p+1}} d\omega_U\right].$$

The connecting homomorphism δ for smooth singular cohomology is the dual of that given in smooth singular homology, and so if $[\alpha] \in H^p\left(U\cap V;\mathbb{R}\right)$, then $\delta\left[\alpha\right]\left(\left[\sigma^{p+1}\right]\right) = \alpha\left(\partial_*\left[\sigma^{p+1}\right]\right)$. And so, $\sigma^{p+1} \in C^\infty_{p+1}\left(M\right)$ can be written as $\sigma^{p+1}_U + \sigma^{p+1}_V$ where σ_U and σ_V are chains and so $\delta\left[\alpha\right]\left(\left[\sigma^{p+1}\right]\right) = \alpha\left[\partial\sigma^{p+1}_U\right]$, where

$\partial\sigma_U^{p+1} = -\partial\sigma_V^{p+1}$ are supported in $U \cap V$. Thus, we have that

$$\begin{aligned}\delta J\left[\omega\right] &= \delta\left[\sigma^p \to \int_{\sigma^p}\omega\right] \\ &= \left[\sigma^{p+1} \to \int_{\partial\sigma_U^{p+1}}\omega\right].\end{aligned}$$

Now we look to see that for $\left[\sigma^{p+1}\right] \in H_{p+1}\left(M\right)$???

$$\begin{aligned}\int_{\sigma^{p+1}} d\omega_U &= \int_{\sigma_U^{p+1}+\sigma_V^{p+1}} d\omega_U = \int_{\sigma_U^{p+1}} d\omega_U + \int_{\sigma_V^{p+1}} d\omega_V \\ &= \int_{\partial\sigma_U^{p+1}} \omega_U + \int_{\partial\sigma_V^{p+1}} \omega_V = \int_{\partial\sigma_U^{p+1}} \omega_U - \int_{\partial\sigma_U^{p+1}} \omega_V = \int_{\partial\sigma_U^{p+1}} \omega,\end{aligned}$$

and so we are done. □

Theorem 30 (de Rham)**.** *For every smooth manifold M and every nonnegative integer p, the de Rham homomorphism $J : H_{dR}^p\left(M\right) \to H^p\left(M;\mathbb{R}\right)$ is an isomorphism.*

Proof. The idea is that we prove it for open subsets of $\mathbb{R}^n$ and then patch together. To do this, we need to show it is true for coordinate balls and their intersections. We know that for convex subsets of $\mathbb{R}^n$, that $H_{dR}^p\left(M\right) = 0 = H^p\left(M;\mathbb{R}\right)$ if $p > 0$, and $H_{dR}^0\left(M\right)$ is generated by the functions that are constant on each connected component. We also know that $H_0\left(M\right)$ is generated by points in each connected component, and it follows that the de Rham homomorphism takes constant functions to the element that assigns the point generating $H_0\left(M\right)$ the value of the constant function on its component.

However, intersections of coordinate balls are not necessarily convex, so we need something stronger. Let's call any manifold M such that $J : H_{dR}^p\left(M\right) \to H^p\left(M;\mathbb{R}\right)$ is an isomorphism a *de Rham manifold*. We will show that any manifold M with a basis for its topology such that every element in the basis and every finite intersection of basis elements is de Rham is itself de Rham. We will call such a basis a *de Rham basis*. Since coordinate balls are convex in their charts, they are de Rham, so we need only show that their intersections are de Rham, which means we need only show that any open set in $\mathbb{R}^n$ is de Rham. This is the trickiest part of the proof.

First, let's show that if we have a finite open cover $\left\{U_i\right\}_{i=1}^k$ of M such that each set and each intersection of sets is de Rham (we call this a *finite de Rham cover*), then M is de Rham. We do this inductively using Mayer-Vietoris. Clearly U_0 is de Rham (by assumption). Suppose $M_j = \cup_{i=1}^j U_i$ is de Rham. We consider the Mayer-Vietoris sequence, with naturality:

$$\begin{array}{ccccccc} H_{dR}^{p-1}\left(M_j\right) \oplus H_{dR}^{p-1}\left(U_{j+1}\right) & \xrightarrow{i^*-j^*} & H_{dR}^{p-1}\left(M_j \cap U_{j+1}\right) & \xrightarrow{\delta} & H_{dR}^p\left(M_j \cup U_{j+1}\right) & \xrightarrow{k^*\oplus\ell^*} & H_{dR}^p\left(M_j\right) \oplus \\ \downarrow J & & \downarrow J & & \downarrow J & & \\ H^{p-1}\left(M_j;\mathbb{R}\right) \oplus H^{p-1}\left(U_{j+1};\mathbb{R}\right) & \xrightarrow{i^*-j^*} & H^{p-1}\left(M_j \cap U_{j+1};\mathbb{R}\right) & \xrightarrow{\delta} & H^p\left(M_j \cup U_{j+1};\mathbb{R}\right) & \xrightarrow{k^*\oplus\ell^*} & H^p\left(M_j;\mathbb{R}\right) \oplus \end{array}$$

The four outer J maps are isomorphisms by assumption, so by the Five Lemma, so is the middle one.

Now we show that if a manifold M has a de Rham basis, then it is de Rham. Suppose M has a de Rham basis. We can find an exhaustion function Φ on M, and so for each $j \in \mathbb{N}$, the set A_j such that $j \leq \Phi \leq j+1$ is compact, and so it has a finite de Rham cover (a finite subcover of the de Rham basis). We can also construct de Rham covers on A'_j, the set such that $j - \frac{1}{2} \leq \Phi \leq j + \frac{3}{2}$, and by using intersections, we can make the covers for A_j such that they are inside A'_j. If we let B_j be the union of the open sets in the de Rham cover for A_j, we have that B_j is open, has a finite de Rham cover, and that $B_j \cap B_\ell$ is empty unless $\ell = j+1$ or $\ell = j-1$. Since the cohomology splits into direct products for disjoint unions, it is clear that any disjoint union of de Rham manifolds is de Rham, thus we have a finite de Rham cover for M given by

$$A = \bigcup_{j \text{ odd}} A_j$$
$$B = \bigcup_{j \text{ even}} A_j$$

(it is clear that the intersection is a disjoint union of de Rham manifolds) and so $M = A \cup B$ is de Rham.

It follows that every open set in $\mathbb{R}^n$ is de Rham, since it has a basis of balls, and intersections of balls are convex. Furthermore, every manifold has a de Rham basis, and is thus de Rham. □

8. Poincaré duality

Theorem 31 (Poincare duality). *If M is a compact, orientable smooth n-manifold, then $\dim H^p_{dR}(M) = \dim H^{n-p}_{dR}(M)$. This comes from the Poincaré duality map*

$$PD : \mathcal{A}^p(M) \to \mathcal{A}^{n-p}(M)^*$$

given by

$$PD(\omega) = \left\{ \eta \to \int_M \omega \wedge \eta \right\}.$$

Note that to prove the theorem, we need compactly supported cohomology groups $H^p_c(M)$, generated by the chain complex of compactly supported forms, and the Poincaré duality map for an arbitrary (possibly noncompact) manifold is

$$PD : \mathcal{A}^p(M) \to \mathcal{A}^{n-p}_c(M)^*$$

given by the same formula.

Example 1. *We have seen that $H^n_{dR}(S^n) \cong H^0_{dR}(S^n)$ and all others groups are zero.*

Example 2. *For any orientable surface Σ, we see that $H^2_{dR}(\Sigma) \cong \mathbb{R} \cong H^0_{dR}(\Sigma)$, and the Poincaré duality theorem does not give a restriction on the middle cohomology group $H^1_{dR}(\Sigma)$.*

Corollary 32. *The Euler characteristic for an odd-dimensional closed, orientable manifold is zero. (Thus the Euler characteristic is not a very meaningful invariant for, say, 3-dimensional manifolds.)*