HOMOLOGY

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1. Introduction

We will explore both simplicial and singular homology.

2. Δ-Complexes

We can write the torus $S^1 \times S^1$, $\mathbb{RP}^2$, and the Klein bottle K^2 as two triangles with their edges identified (draw picture). For simplicial homology, we will construct spaces by identifying generalized triangles called simplices.

Definition 1. *An n-dimensional* simplex *(or n-simplex) $\sigma^n = [v_0, v_1, \ldots, v_n]$ is the smallest convex set in a Euclidean space $\mathbb{R}^m$ containing the $n+1$ points $v_0, \ldots, v_n$. We usually specify that for an n-simplex, we have that the points are not contained in any hyperplane of dimension less than n. The* standard n-simplex *is*

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i = 0, \ldots, n \right\}.$$

Definition 2. *We call 1-simplices* vertices*, and 2-simplices* edges*. Given a simplex σ^n, any $(n-1)$-dimensional subsimplex is called a* face.

It will be important to keep track of the ordering of the vertices, so $[v_0, v_1] \neq [v_1, v_0]$. Note that given any n-simplex, it induces an ordering on smaller simplices. For instance, an edge can be written $[v_i, v_j]$ where $i < j$. *For any sub-simplex, we can use the same ordering as in the larger simplex.*

Once we have specified the ordering to the vertices in σ^n, we have a natural linear transformation of the standard n-simplex to another σ^n, i.e.,

$$\sigma^n = \left\{ \sum_{i=1}^{n} t_i v_i : \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i = 0, \ldots, n \right\}$$

so the map is $(t_0, \ldots, t_n) \to \sum_{i=1}^{n} t_i v_i$. This map gives the barycentric coordinates for points in the simplex.

Definition 3. *The union of all of the faces of Δ^n is called the* boundary *of Δ^n, and is denoted as $\partial\Delta^n$. (If $n = 0$, then the boundary is empty.) The open simplex is interior of Δ^n, i.e., $\mathring{\Delta}^n = \Delta \setminus \partial\Delta$*

Definition 4. *A* Δ-complex *structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with n depending on α, such that*

(1) *The restriction of $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective, and each point of X is in the image of exactly one such restriction.*

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(2) *Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. We identify the faces of Δ^n with Δ^{n-1} by the canonical linear homeomorphism that preserves the ordering of the vertices.*
(3) *A set $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.*

Note that (3) give shows how to give a topology to a quotient of disjoint simplices identifies according to (2). These can be built inductively on n by attaching new simplices (and possibly identifying sub-simplices.

Remark 1. *Not all spaces admit a Δ-complex structure. In particular, the space X must be Hausdorff.*

By the construction, the set X is a disjoint union of open simplices e_α^n of various dimensions. The open simplex is the homeomorphic image $\sigma_\alpha^n\left(\mathring{\Delta}^n\right)$ of $\mathring{\Delta}^n$, and we call the map σ_α a *characteristic map.*

3. Simplicial homology

Definition 5. *Suppose X has a Δ-complex structure. Let $\Delta_n(X)$ denote the free abelian group generated by the open n-simplices e_α^n of X. Elements of $\Delta_n(X)$ are called n-chains, and can be written as a finite formal sum*

$$\sum_\alpha c_\alpha e_\alpha^n$$

with $c_\alpha \in \mathbb{Z}$.

Remark 2. *We could also replace e_α^n with the characteristic maps σ_α^n and consider these as chains.*

We can consider the boundary of a simplex as a chain:

$$\begin{aligned}
\partial[v_0, v_1] &= [v_1] - [v_0] \\
\partial[v_0, v_1, v_2] &= [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \\
\partial[v_0, v_1, v_2, v_3] &= [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]
\end{aligned}$$

This allows us to define a boundary homomorphism:

Definition 6. *For a Δ-complex X, the boundary homomorphism $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ is generated by*

$$\partial_n(\sigma_\alpha^n) = \sum_{j=0}^{n} (-1)^j \left.\sigma_\alpha^n\right|_{[v_0, v_1, \ldots, \hat{v}_j, \ldots, v_n]}.$$

Lemma 7. *The composition $\Delta_n(X) \overset{\partial_n}{\to} \Delta_{n-1}(X) \overset{\partial_{n-1}}{\to} \Delta_{n-2}(X)$ is zero.*

Proof. We compute

$$\partial_{n-1} \left.\sigma\right|_{[v_0, v_1, \ldots, \hat{v}_j, \ldots, v_n]} = \sum_{i=0}^{j-1} (-1)^i \left.\sigma\right|_{[v_0, v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n]} + \sum_{i=j+1}^{n} (-1)^{i-1} \left.\sigma\right|_{[v_0, v_1, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n]}$$

$$\begin{aligned}\partial_{n-1}\partial_n(\sigma) &= \partial_{n-1}\left(\sum_{j=0}^{n}(-1)^j\ \sigma|_{[v_0,v_1,\dots,\hat{v}_j,\dots,v_n]}\right)\\ &= \sum_{j=0}^{n}\sum_{i=0}^{j-1}(-1)^{i+j}\ \sigma|_{[v_0,v_1,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]}+\sum_{j=0}^{n}\sum_{i=j+1}^{n}(-1)^{i+j-1}\ \sigma|_{[v_0,v_1,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]}\\ &= \sum_{i<j}(-1)^{i+j}\ \sigma|_{[v_0,v_1,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]}+\sum_{j<i}(-1)^{i+j-1}\ \sigma|_{[v_0,v_1,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]}\\ &= 0\end{aligned}$$

since we can switch the roles of i and j in the second sum. □

Definition 8. *The simplicial homology groups are defined as* $H_n^{\Delta}(X) = \ker\partial_n / \operatorname{Im}\partial_{n+1}$. *This makes sense since* $\partial_{n+1}\partial_n = 0$ *implies that* $\operatorname{Im}\partial_{n+1} \subseteq \ker\partial_n$.

Definition 9. *Elements of* $\ker\partial$ *are called* cycles*. Elements of* $\operatorname{Im}\partial$ *are called* boundaries*. One often sees the notation* $Z_n(X) = \ker\partial_n$ *and* $B_n(X) = \operatorname{Im}\partial_{n+1}$*, and so homology is written*

$$H_n(X) = Z_n(X)/B_n(X).$$

The great thing about simplicial homology is that it can be computed reasonably easily given a Δ-complex structure on X.

Example 1. *Consider the torus given as*

$$\begin{aligned}\Delta_2\left(S^1\times S^1\right) &= \left\langle\sigma_1^2,\sigma_2^2\right\rangle\\ \Delta_1\left(S^1\times S^1\right) &= \left\langle\sigma_1^1,\sigma_2^1,\sigma_3^1\right\rangle\\ \Delta_0\left(S^1\times S^1\right) &= \left\langle\sigma_1^0\right\rangle\end{aligned}$$

such that

$$\begin{aligned}\partial_2\sigma_1^2 &= \sigma_1^1-\sigma_2^1+\sigma_3^1\\ \partial_2\sigma_2^2 &= -\sigma_1^1+\sigma_2^1-\sigma_3^1\end{aligned}$$

and all other maps are zero. Thus

$$\begin{aligned}H_2^{\Delta}(X) &= \left\langle\sigma_1^2+\sigma_2^2\right\rangle \cong \mathbb{Z}\\ H_1^{\Delta}(X) &= \left\langle\sigma_1^1,\sigma_2^1,\sigma_3^1\right\rangle/\left\langle\sigma_1^1-\sigma_2^1+\sigma_3^1,-\sigma_1^1+\sigma_2^1-\sigma_3^1\right\rangle \cong \mathbb{Z}\times\mathbb{Z}\\ H_0^{\Delta}(X) &= \left\langle\sigma_0^1\right\rangle \cong \mathbb{Z}\end{aligned}$$

4. Singular homology

Definition 10. *A* singular n-simplex *in a space* X *is a (continuous) map* $\sigma : \Delta^n \to X$*. The group of singular* n*-chains* $C_n(X)$ *is the free abelian group generated by singular* n*-simplices on* X*, so elements are finite sums form*

$$\sum_i c_i\sigma_i^n.$$

The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ *is defined by*

$$\partial_n\sigma = \sum_{i=0}^{n}(-1)^i\ \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}.$$

Note that implicit in this definition is the identification of $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ with Δ^{n-1}. Also note that $\partial_{n-1}\partial_n = 0$ by the same proof as with Δ-complexes. Thus it is reasonable to define the homology theory.

Definition 11. *The* singular homology groups *are defined as*

$$H_n(X) = \ker \partial_n / \operatorname{Im} \partial_{n+1}.$$

Here are some comments about singular homology groups:

- It is clear that homeomorphic spaces have isomorphic singular homology groups (not clear for Δ-complexes).
- The chain groups are enormous, usually uncountable. It is not clear that if X is a Δ-complex with finitely many simplices that the homology is finitely generated or that $H_n(X) = 0$ for n larger than the largest dimensional simplex in the Δ-complex (both trivial for simplicial homology).

Remark 3. *There is a construction in Hatcher showing that singular homology is actually an instance of simplicial homology, though the simplicial chains are generally uncountably generated.*

Remark 4. *Hatcher also describes a geometric way to think about homology in terms of images of manifolds.*

Proposition 12. *If X is a space with path components X_α, there is an isomorphism of $H_n(X)$ with the direct sum $\bigoplus_\alpha H_n(X_\alpha)$.*

Proof. Since the image of a simplex is connected, each simplex lies on one path component, and hence $C_n(X)$ splits into a direct sum. Since the boundaries are in the same component, we get that the boundary splits into $C_n(X_\alpha) \to C_{n-1}(X_\alpha)$, and so do the kernel and images, so this becomes an isomorphism on homology. □

Proposition 13. *If X is nonempty and path connected, then $H_0(X) \cong \mathbb{Z}$. Hence for any space, $H_0(X)$ is isomorphic to a direct sum of $\mathbb{Z}$'s, one for each path component.*

Proof. Define the homomorphism

$$\varepsilon : C_0(X) \to \mathbb{Z}$$

by

$$\varepsilon\left(\sum_i c_i \sigma_i\right) = \sum_i c_i.$$

If X is nonempty, then clearly this is surjective. We claim that $\ker \varepsilon = \operatorname{Im} \partial_1$, which would imply that

$$H_0(X) = C_0(X) / \operatorname{Im} \partial_1 = C_0(X) / \ker \varepsilon \cong \mathbb{Z}.$$

If $\sum_i c_i \sigma_i \in \operatorname{Im} \partial_1$, then

Elements of $\operatorname{Im} \partial_1$ are integer linear combinations of elements of the form $[v_i] - [v_j]$, and ε of these is zero, so $\operatorname{Im} \partial_1 \subseteq \ker \varepsilon$. Now given a chain in $\ker \varepsilon$, it can be written as a sum $\sum ([v_i] - [v_j])$, and each of these can be realized as the boundary of a path from v_i to v_j since both are path components. Thus $\ker \varepsilon \subseteq \operatorname{Im} \partial_1$. □

Proposition 14. $H_n(pt) = 0$ *for* $n > 0$ *and* $H_0(pt) \cong \mathbb{Z}$.

Proof. We know the last statement since a point has one path component. Each chain complex $C_n(pt)$ is generated by one simplex. Since the boundary gives

$$\partial\sigma^n = \sum_{i=0}^{n} (-1)^j \sigma^{n-1},$$

we have that $\partial = 0$ if n is odd and the identity if n is even. Thus we get the chain complex

$$\cdots \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

The result on homology follows. $\square$

Definition 15. *The reduced homology groups $\tilde{H}_n(X)$ are the homology groups of the augmented chain complex*

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

We have already seen that $\varepsilon\partial_1 = 0$, we have that ε induces a map $H_0(X) \to \mathbb{Z}$, and the kernel of this map is precisely $\tilde{H}_0(X)$. It follows that

$$H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$$
$$H_n(X) \cong \tilde{H}_n(X) \text{ if } n > 0.$$

5. Homotopy invariance

While it is clear that homology groups are invariant under homeomorphism, it is also true that it is invariant under homotopy equivalence. We first look at induced homomorphisms.

Definition 16. *Given a continuous map $f : X \to Y$, there is an induced map $f_\# : C_n(X) \to C_n(Y)$ given by*

$$f_\#(\sigma^n) = f \circ \sigma^n$$

and extended linearly to chains.

Proposition 17. $f_\#\partial = \partial f_\#$.

Proof.

$$\begin{aligned} f_\#\partial(\sigma^n) &= f_\# \sum_{j=0}^{n} (-1)^j \sigma^n|_{[v_0,v_1,\ldots,\hat{v}_j,\ldots,v_n]} \\ &= \sum_{j=0}^{n} (-1)^j (f \circ \sigma^n)|_{[v_0,v_1,\ldots,\hat{v}_j,\ldots,v_n]} \\ &= \partial f_\#(\sigma^n). \end{aligned}$$

$\square$

Definition 18. *A map $\phi : C_n(X) \to C_n(Y)$ satisfying $\phi\partial = \partial\phi$ is called a* chain map.

This gives us a commutative diagram of chain complexes (draw picture). It follows that if $\alpha \in \ker\partial : C_n(X) \to C_{n-1}(X)$ then

$$\partial f_\#(\alpha) = f_\#\partial(\alpha) = 0$$

and so the kernel maps to the kernel, while if $\beta = \partial\gamma$ then

$$f_\# \beta = f_\# \partial\gamma = \partial f_\# \gamma$$

and so the image of ∂ maps to the image of ∂.

It follows that a chain map induces a map on homology, $f_* : H_n(X) \to H_n(Y)$ and similarly for reduced homology.

Proposition 19. *A chain map between complexes induces a homomorphism between homology groups of the complexes.*

Proposition 20. *The following are true:*

1. $(f \circ g)_* = f_* g_*$ *(where $f : X \to Y$ and $g : Y \to Z$).*
2. $Id_* = Id$.

Proof. These follow easily from associativity of composition of maps. □

Theorem 21. *If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$.*

Corollary 22. *Maps $f_* : H_n(X) \to H_n(Y)$ induced by a homotopy equivalence $f : X \to Y$ are isomorphisms for all n.*

The theorem is proven by two propositions:

Proposition 23. *A homotopy between f and g induces a chain homotopy $P : C_n(X) \to C_{n+1}(Y)$.*

Definition 24. *A* chain homotopy *between maps $f_\#$ and $g_\#$ is a map $P : C_n(X) \to C_{n+1}(Y)$ satisfying*

$$\partial P + P\partial = g_\# - f_\#.$$

Two maps are chain homotopic *if there exists a chain homotopy between them.*

Proposition 25. *Chain homotopic maps induce the same map on homology.*

Proof. If $\alpha \in C_n(X)$ is a cycle, then $g_\# \alpha - f_\# \alpha = \partial P \alpha$, and so $g_*[\alpha] = f_*[\alpha]$. □

The key is constructing the chain homotopy, which is a prism operators P. The homotopy $F : X \times I \to Y$ gives, for each simplex $\sigma^n : \Delta^n \to X$ a map

$$F \circ (\sigma \times 1) : \Delta^n \times I \to Y.$$

$\Delta^n \times I$ can be broken up into $(n+1)$-simplices in the following way (see Hatcher for the details): if we let $[v_0, \ldots, v_n]$ be the vertices for $\Delta^n \times \{0\}$ and $[w_0, \ldots, w_n]$ be the vertices for $\Delta^n \times \{1\}$, we have simplices

$$\sigma_i^{n+1} = [v_0, \ldots, v_i, w_i, \ldots., w_n].$$

(Technically, a simplex is a map, so we want $\sigma \times 1$ restricted to this.) We can construct the prism operator

$$P : C_n(X) \to C_{n+1}(Y)$$

by

$$\begin{aligned} P(\sigma^n) &= \sum_{i=0}^{n} (-1)^i F \circ \sigma_i^{n+1} \\ &= \sum_{i=0}^{n} (-1)^i \left[F \circ (\sigma \times 1)\right]|_{[v_0, \ldots, v_i, w_i, \ldots., w_n]}. \end{aligned}$$

We can now compute

$$\begin{aligned}\partial P(\sigma) &= \sum_{j\leq i}(-1)^{i+j}\left[F\circ(\sigma\times 1)\right]|_{[v_0,\ldots,\hat{v}_j,\ldots v_i,w_i,\ldots,w_n]}\\ &\quad+\sum_{j\geq i}(-1)^{i+j+1}\left[F\circ(\sigma\times 1)\right]|_{[v_0,\ldots,v_i,w_i,\ldots\hat{w}_j,\ldots,w_n]}.\end{aligned}$$

Notice that in the top term, $i=j=a$ cancels with the bottom term $i=j=a-1$, as long as $a\neq 0$ and $a\neq n$. In those cases we get

$$\begin{aligned}F\circ(\sigma\times\{1\}) &= g_{\#}\sigma\\ -F\circ(\sigma\times\{0\}) &= -f_{\#}\sigma\end{aligned}$$

Also compute

$$\begin{aligned}P\partial(\sigma) &= P\sum_{j=1}^{n}(-1)^{j}\,\sigma|_{[v_0,\ldots,\hat{v}_j,\ldots,v_n]}\\ &= \sum_{i<j}(-1)^{i+j+1}\left[F\circ(\sigma\times 1)\right]|_{[v_0,\ldots,\hat{v}_j,\ldots,v_i,w_i,\ldots,w_n]}\\ &\quad+\sum_{i>j}(-1)^{i+j}\left[F\circ(\sigma\times 1)\right]|_{[v_0,\ldots,v_i,w_i,\ldots,\hat{w}_j,\ldots,w_n]}.\end{aligned}$$

So we get

$$\partial P = -P\partial + g_{\#}\sigma - f_{\#}\sigma.$$

Thus the map P is a chain homotopy and so we get that $g_* = f_*$.

6. Relative homology and the exact sequence of the pair

In studying quotients and other spaces, it will be extremely useful to consider relative homology.

Definition 26. *Suppose X is a topological space and A is a subspace of X. Let the relative chains $C_n(X,A)$ be defined by*

$$C_n(X,A) = C_n(X)/C_n(A).$$

Define the boundary maps to be

$$\partial_n : C_n(X,A)\to C_{n-1}(X,A)$$

defined by the induced quotient maps.

We should check that the boundary makes sense, i.e., that $\partial_n : C_n(X)\to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, but this is true since A is a subspace! Furthermore, it is clear that $\partial^2 = 0$, so there is a homology theory for this chain complex.

Definition 27. *The relative homology groups $H_n(X,A)$ are the homology groups of the relative chains.*

Here are some observations:

- Elements of $H_n(X,A)$ are represented by relative cycles: n-chains $\alpha\in C_n(X)$ such that $\partial\alpha\in C_{n-1}(A)$.
- A relative cycle α in $H_n(X,A)$ is trivial if it is a relative boundary: $\alpha = \partial\beta+\gamma$ where $\beta\in C_{n+1}(X)$ and $\gamma\in C_n(A)$.

For this reason, we can almost think of $H_n(X, A)$ as the homology of the quotient X/A, though not quite!

The main theorem of this section is that the relative homology groups form an exact sequence with the homology groups of X and A.

Definition 28. *A sequence of homomorphisms*

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_{n+1}} A_{n-1} \longrightarrow \cdots$$

is exact *if* $\ker \alpha_n = \operatorname{Im} \alpha_{n-1}$ *for all* n.

Note that an exact sequence is a chain complex (since $\operatorname{Im} \alpha \subseteq \ker \alpha$) that has trivial homology (since $\ker \alpha \subseteq \operatorname{Im} \alpha$). Note the following:

- $0 \longrightarrow A \xrightarrow{\alpha} B$ is exact iff α if injective.
- $A \xrightarrow{\alpha} B \longrightarrow 0$ is exact iff α is surjective.
- $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$ is exact iff α is an isomorphism.
- If $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is exact, it is called a short exact sequence, and α is injective, β is surjective, and $\ker \beta = \operatorname{Im} \alpha$.

The key facts are the following:

Proposition 29. *A short exact sequence of chain complexes gives a long exact sequence of homology.*

Definition 30. *A short exact sequence of chain complexes* $0 \longrightarrow A_n \xrightarrow{\alpha} B_n \xrightarrow{\beta} C_n \longrightarrow 0$ *is a collection of chain complexes such that the following diagrams are commutative for each* n*:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_n & \xrightarrow{\alpha_n} & B_n & \xrightarrow{\beta_n} & C_n & \longrightarrow & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \longrightarrow & A_{n-1} & \xrightarrow{\alpha_{n-1}} & B_{n-1} & \xrightarrow{\beta_{n-1}} & C_{n-1} & \longrightarrow & 0 \end{array} \tag{6.1}$$

Proposition 31. *There is a short exact sequence of chain complexes:*

$$0 \longrightarrow C_n(A) \xrightarrow{\iota_\#} C_n(X) \xrightarrow{q_\#} C_n(X, A) \longrightarrow 0$$

where $\iota : A \to X$ *is the inclusion map and* $q : C_n(X) \to C_n(X)/C_n(A) = C_n(X, A)$ *is the quotient map.*

The proof of Proposition 29 is by a method of proof called diagram chasing.

Proof of Proposition 29. The main difficulty is defining the boundary map $\partial_* : H_n(C) \to H_{n-1}(A)$. This can be done as follows. Let $c \in C_n(C)$ be a cycle. Since it is a cycle, we have $\partial c = 0$. Since β is surjective, there is a chain $b \in B_n$ such that $\beta b = c$. By commutativity, we have that

$$0 = \partial c = \partial \beta b = \beta \partial b.$$

It follows that $\partial b \in \ker \beta = \operatorname{Im} \alpha$, and so there is a unique $a \in A_{n-1}$ such that $\alpha a = \partial b$. Now we need to see that it is a cycle, but that follows since

$$0 = \partial \alpha a = \alpha \partial a,$$

but since α is injective, we must have $\partial a = 0$.

This gives a 'map' from $Z_n(C)$ to $Z_{n-1}(A)$, but what if we chose a different b' such that $\beta b' = c$, and therefore got a'? Then $\beta(b' - b) = 0$, and so $b' - b \in \ker \beta =$

$\operatorname{Im}\alpha$, so there is $a'' \in A_n$ such that $\alpha a'' = b' - b$. Note that $\partial b - \partial b' = \partial\alpha a'' = \alpha\partial a''$. Notice that

$$\alpha\left(a - a'\right) = \partial b - \partial b',$$

and since α is injective, we have $a - a' = \partial a''$. Thus there is a well-defined map $Z_n\left(C\right) \to H_{n-1}\left(A\right).$

Now consider the image of $\partial c' \in Z_n\left(C\right)$. Following the map construction, we have that there is a chain $b \in B_n$ such that $\beta b = \partial c'$. However, there must also be $b' \in B_{n+1}$ such that $\beta b' = c'$, and it must satisfy $\beta\partial b' = \partial\beta b' = \partial c'$. Thus we can choose $b = \partial b'$. It follows that $\partial b = 0$ and so the corresponding $a = 0$. Thus the map is well-defined $H_n\left(C\right) \to H_{n-1}\left(A\right)$. Note that the map works like this: given $[c] \in H_n\left(C\right)$, there is a $b \in B_n$ such that $\beta b = c$ and $a \in A_{n-1}$ such that $\alpha a = \partial b$, and the map is $\partial_*[c] = [a]$.

Now we need to prove that the long sequence is exact:

$$\cdots \longrightarrow H_{n+1}\left(C\right) \xrightarrow{\partial_*} H_n\left(A\right) \xrightarrow{\alpha_*} H_n\left(B\right) \xrightarrow{\beta_*} H_n\left(C\right) \xrightarrow{\partial_*} H_{n-1}\left(A\right) \longrightarrow \cdots$$

- $\operatorname{Im}\alpha_* = \ker\beta_*$: suppose $[b] \in \operatorname{Im}\alpha_*$, so there exist $a \in Z_n\left(A\right)$ such that $\alpha a = b$. We know that $\beta b = 0$ by exactness of the short exact sequence, and so $\beta_*[b] = 0$. The reverse inclusion is similar.
- $\operatorname{Im}\beta_* = \ker\partial_*$: If $[c]$ is in the kernel of ∂_*, then there is a $b \in B_n$ and $a' \in A_n$ such that $\beta b = c$ and $\alpha\partial a' = \partial b$. But also $\partial\alpha a' = \partial b$, and $\beta\alpha a' = 0$ so $\beta\left(b - \alpha a'\right) = c$ and $\partial\left(b - \alpha a'\right) = 0$, so $b - \alpha a'$ is a cycle and $[c] = \beta_*\left[b - \alpha a'\right]$. Conversely, if $[c] = \beta_*[b]$, then $\partial b = 0$ and $\beta b = c$, so $\partial_*[c] = 0$ (actually, we have $\beta b = c + \partial c'$, but then there is a b' such that $\beta b' = c'$ and $\beta\left(b + \partial b'\right) = c + \partial c'$ and $b + \partial b'$ is still a cycle).
- $\operatorname{Im}\partial_* = \ker\alpha_*$: If $[a]$ is in the kernel of α_* then $\alpha a = \partial b'$ for some cycle $b' \in B_{n+1}$. Since $\partial b' = 0$, it follows that $\partial\beta b' = \beta\partial b' = 0$, and so $[a] = \partial_*[\beta b']$. Conversely, if $[a] = \partial_*[c]$, then $\alpha a = \partial b$ such that $\beta b = c$. It follows that $\alpha_*[a] = 0$.

□

So the long exact sequence of a pair (X, A) is

$$\cdots \longrightarrow H_{n+1}\left(X, A\right) \xrightarrow{\partial_*} H_n\left(A\right) \xrightarrow{\iota_*} H_n\left(X\right) \xrightarrow{q_*} H_n\left(X, A\right) \xrightarrow{\partial_*} H_{n-1}\left(A\right) \longrightarrow \cdots$$

Note that the boundary map ∂_* is quite explicit in this case: suppose $[\xi] \in H_n\left(X, A\right)$. Then it is represented by a chain $\xi \in C_n\left(X\right)$ such that $\partial\xi \in C_{n-1}\left(A\right)$. Furthermore, since $\partial^2\xi = 0$, in face, $[\partial\xi]$ represents a class in $H_{n-1}\left(A\right)$, so the map really is induced by ∂.

Proposition 32. *There is a long exact sequence of reduced homology groups. Furthermore, $\tilde{H}_n\left(X, A\right) \cong H_n\left(X, A\right)$ for all n if A is nonempty.*

Proof. We augment the short exact sequence of chain complexes

$$0 \to C_n\left(A\right) \to C_n\left(X\right) \to C_n\left(X\right)/C_n\left(A\right) \to 0$$

where $n \geq 0$ by

$$0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \to 0$$

in the $n = -1$ place (where the first two boundary maps are ε), and the nontrivial map above is the identity. The same construction holds. Note that since we augment the relative complex with zero, the chain complex for reduced relative homology is the same as for relative homology. □

Proposition 33. *If two maps $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs, then $f_* = g_*$.*

Proof. Recall the prism operator $P : C_n(X) \to C_{n+1}(Y)$. If we restrict P to $C_n(A)$, its image will certainly lie inside $C_{n+1}(B)$ (since the homotopy is through maps of pairs) and so there is a map $P : C_n(X, A) \to C_n(Y, B)$. The map P is still a chain homotopy, and the result follows. $\square$

There is also a long exact sequence of a triple derived in a similar way. The triple is $A \subseteq B \subseteq C$ and the necessary inclusions are $(A, B) \to (B, C)$. This gives the long exact sequence

$$\cdots \longrightarrow H_{n+1}(C, B) \xrightarrow{\partial_*} H_n(C, A) \xrightarrow{\alpha_*} H_n(B, A) \xrightarrow{\beta_*} H_n(C, B) \xrightarrow{\partial_*} H_{n-1}(C, A) \longrightarrow \cdots$$

arising from the short exact sequence

$$0 \longrightarrow C_n(C, A) \xrightarrow{\alpha} C_n(B, A) \xrightarrow{\beta} C_n(C, B) \longrightarrow 0$$

which is exact because it is

$$0 \longrightarrow C_n(C)/C_n(A) \xrightarrow{\alpha} C_n(B)/C_n(A) \xrightarrow{\beta} C_n(C)/C_n(B) \longrightarrow 0.$$

7. Excision and quotients

We will not prove the excision theorem because it is a bit technical and we don't have the time. The proof is not particularly difficult, just technical. The main idea is the following proposition.

Let $\mathcal{U} = \{U_j\}$ be a collection of subspaces of X whose interiors form an open cover of X, and let $C_n^{\mathcal{U}}(X)$ be the subgroup of $C_n(X)$ consisting of chains $\sum c_i \sigma_i$ such that σ_i has image contained in some set $U \in \mathcal{U}$. The boundary map takes $C_n^{\mathcal{U}}(X)$ to $C_{n-1}^{\mathcal{U}}(X)$, and so there are homology groups $H_n^{\mathcal{U}}(X)$.

Proposition 34. *The inclusion $\iota : C_n^{\mathcal{U}}(X) \to C_n(X)$ is a chain homotopy equivalence, i.e., there is a chain map $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ such that $\iota\rho$ and $\rho\iota$ are chain homotopic to the identity. Hence ι induces an isomorphism $H_n^{\mathcal{U}}(X) \cong H_n(X)$.*

The main idea is that any chain can be broken up into pieces all in one of the sets in $\mathcal{U}$. This is done by barycentric subdivision of the simplices. We will not prove this proposition.

Theorem 35 (Excision Theorem)**.** *Given subspaces $Z \subseteq A \subseteq X$ such that the closure of Z is contained in the interior of A, then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces isomorphisms*

$$H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$$

for all n. Equivalently, for subspaces $A, B \subseteq X$ whose interiors cover X, the inclusion $(B, A \cap B) \to (X, A)$ induces isomorphisms on homology.

Proof. We will not prove this, but the idea is that the chain homotopy from Proposition 34 can apply to quotients, and then one can pick out pieces we don't want (i.e., chains in Z). $\square$

We can now prove that the relative sequence is true for quotients:

Theorem 36. *If X is a space and A is a nonempty closed subspace that is a deformation retract of some neighborhood in X, then there is an exact sequence*

$$\cdots \longrightarrow \tilde{H}_n(A) \xrightarrow{\iota_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial_*} \tilde{H}_{n-1}(A) \xrightarrow{\iota_*} \tilde{H}_{n-1}(X) \xrightarrow{j_*} \tilde{H}_{n-1}(X/A) \longrightarrow \cdots$$

where ι is the inclusion $A \to X$ and j is the quotient map $X \to X/A$.

Proof. We need the long exact sequence of the triple (X, V, A), where A is a deformation retract of the neighborhood V. We have the following commutative diagram:

$$\begin{array}{ccccc} H_n(X,A) & \longrightarrow & H_n(X,V) & \longleftarrow & H_n(X\setminus A, V\setminus A) \\ \downarrow q_* & & \downarrow q_* & & \downarrow q_* \\ H_n(X/A, A/A) & \longrightarrow & H_n(X/A, V/A) & \longleftarrow & H_n(X/A\setminus A/A, V/A\setminus A/A) \end{array}$$

We have that the rightmost maps are isomorphisms by excision. The upper left map is an isomorphism using the long exact sequence of the triple since $H_n(V,A)$ is trivial since A is a deformation retraction of V, and hence $H_n(V,A) \cong H_n(A,A) \cong 0$. Similarly, we get that the lower left map is an isomorphism. Since q is a homeomorphism on the complement of A, the rightmost vertical map is an isomorphism. By the long exact sequence of the pair (with relative homology), it is clear that $H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$ since $\tilde{H}_n(A/A) \cong 0$. $\square$

Example 2. *We can compute the homology groups of spheres S^n by induction on n. Note that $S^n \approx D^n/\partial D^n$, and $\partial D^n \approx S^{n-1}$ is the deformation retract of a small annulus around the boundary, so by the exact sequence we have*

$$\cdots \longrightarrow \tilde{H}_n\left(S^{m-1}\right) \xrightarrow{\iota_*} \tilde{H}_n(D^m) \xrightarrow{j_*} \tilde{H}_n(S^m) \xrightarrow{\partial_*} \tilde{H}_{n-1}\left(S^{m-1}\right) \xrightarrow{\iota_*} \tilde{H}_{n-1}(D^m) \xrightarrow{j_*} \tilde{H}_{n-1}(S^m) \longrightarrow \cdots$$

and since $\tilde{H}_n(D^m) = 0$ for all n, we have that $\tilde{H}_n(S^m) \cong \tilde{H}_{n-1}\left(S^{m-1}\right)$. We know that

$$\tilde{H}_n\left(S^0\right) \cong \left\{ \begin{array}{c} \mathbb{Z} \text{ if } n = 0 \\ 0 \text{ otherwise} \end{array} \right\}.$$

It follows that

$$\tilde{H}_n(S^m) \cong \left\{ \begin{array}{c} \mathbb{Z} \text{ if } n = m \\ 0 \text{ otherwise} \end{array} \right\}.$$

Proposition 37 (Brouwer Fixed Point Theorem)**.** *D^n is not a deformation retraction of ∂D^n. Hence every map $D^n \to D^n$ has a fixed point.*

Proof. Recall the argument in dimension 2 that if there is a map $f : D^2 \to D^2$ without a fixed point, then there is a retraction $D^2 \to \partial D^2$. The same argument works for D^n. Recall that a retraction satisfies:

$$r \circ \iota = Id_{\partial D^n},$$

and so the composition

$$\tilde{H}_{n-1}(\partial D^n) \xrightarrow{\iota_*} \tilde{H}_{n-1}(D^n) \xrightarrow{r_*} \tilde{H}_{n-1}(\partial D^n)$$

is an isomorphism. But this is impossible since $\tilde{H}_{n-1}(D^n) = 0$ and $\tilde{H}_{n-1}(\partial D^n) \cong \mathbb{Z}$. $\square$

Theorem 38 (Invariance of Dimension)**.** *If $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are nonempty and homeomorphic open sets, then $n = m$.*

Remark 5. *For diffeomorphic, this followed from inverse function theorem.*

Proof. By excision, we have that $H_k(U, U \setminus \{x_0\}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x_0\})$ for a point $x_0 \in U$. From the long exact sequence of the pair, we have that $H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x_0\}) \cong H_k(\mathbb{R}^n \setminus \{x_0\})$, but since $\mathbb{R}^n \setminus \{x_0\}$ is homotopy equivalent to S^{n-1}, we get that that $m = n$ if U and V are homeomorphic. □

8. Naturality

Theorem 39. *The long exact sequence of the pair is natural, i.e., for any map $f : (X, A) \to (Y, B)$, the the following diagram is commutative:*

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow & H_{n+1}(X,A) & \xrightarrow{\partial_*} & H_n(A) & \xrightarrow{\iota_*} & H_n(X) & \xrightarrow{q_*} & H_n(X,A) & \xrightarrow{\partial_*} & H_{n-1}(A) & \longrightarrow \cdots \\ & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\ \cdots \longrightarrow & H_{n+1}(Y,B) & \xrightarrow{\partial_*} & H_n(B) & \xrightarrow{\iota_*} & H_n(Y) & \xrightarrow{q_*} & H_n(Y,B) & \xrightarrow{\partial_*} & H_{n-1}(B) & \longrightarrow \cdots \end{array}$$

In fact, naturality follows for any long exact sequence arising from a short exact sequence of chain complexes.

Theorem 40. *Given two short exact sequences of chain complexes (6.1) and chain maps ϕ, ψ, ξ between them,i.e.,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_n & \xrightarrow{\alpha_n} & B_n & \xrightarrow{\beta_n} & C_n & \longrightarrow & 0 \\ & & \downarrow \phi & & \downarrow \psi & & \downarrow \xi & & \\ 0 & \longrightarrow & D_n & \xrightarrow{\delta_n} & E_n & \xrightarrow{\varepsilon_n} & F_n & \longrightarrow & 0 \end{array}$$

is commutative, the following diagram is commutative:

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow & H_{n+1}(C) & \xrightarrow{\partial_*} & H_n(A) & \xrightarrow{\alpha_*} & H_n(B) & \xrightarrow{\beta_*} & H_n(C) & \xrightarrow{\partial_*} & H_{n-1}(A) & \longrightarrow \cdots \\ & \downarrow \xi_* & & \downarrow \phi_* & & \downarrow \psi_* & & \downarrow \xi_* & & \downarrow \phi_* & \\ \cdots \longrightarrow & H_{n+1}(F) & \xrightarrow{\partial_*} & H_n(D) & \xrightarrow{\delta_*} & H_n(E) & \xrightarrow{\varepsilon_*} & H_n(F) & \xrightarrow{\partial_*} & H_{n-1}(D) & \longrightarrow \cdots \end{array}$$

Proof. Since $\psi\alpha = \phi\delta$ and $\xi\beta = \varepsilon\psi$, two of the squares are easily seen to be commutative (since each is a chain map). For the last square, we check $\phi_*\partial_*[c] = \phi_*[a] = [\phi(a)]$, where there is a chain $b \in B_n$ such that $\alpha a = \partial b$ and $\beta b = c$.

Also $\partial_*\xi_*[c] = [d]$ where there is a chain $e \in E_n$ such that $\varepsilon e = \xi_*[c] = [\xi(c)]$ and $\delta d = \partial e$. The claim is that $[d] = [\phi(a)]$.

Note that we have a choice of chains b and e. Suppose we take $e = \psi(b)$. Then

$$\begin{aligned} \varepsilon e &= \varepsilon\psi(b) = \xi\beta(b) = \xi(c) \\ \delta\phi(a) &= \psi\alpha(a) = \psi\partial b = \partial\psi b = \partial e. \end{aligned}$$

It follows that $\partial_*[\xi(c)] = [\phi(a)]$, i.e., $\partial_*\xi_*[c] = \phi_*\partial_*[c]$. □

9. Equivalence of simplicial and singular homology

Note that if X is a Δ-complex, then simplicial chains are singular chains. Thus there is an inclusion $\Delta_n(X) \to C_n(X)$, and since boundaries are respected, it induces a homomorphism on homology.

Theorem 41. *The homomorphisms $H_n^{\Delta}(X) \to H_n(X)$ are isomorphisms for all n and Δ-complexes X.*

Remark 6. *This can be done more generally with relative homology, but we will only prove this.*

Lemma 42 (The Five-Lemma)**.** *In a commutative diagram of abelian groups as follows:*

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{\ell} & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \varepsilon \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{\ell'} & E' \end{array}$$

if the two rows are exact and $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, then so is γ.

Remark 7. *In fact, it can be proven that γ is surjective if β and δ are surjective and ε is injective, and that γ is injective if β and δ are injective and α is surjective.*

Proof. The proof is a diagram chase. First we show that γ is surjective. Given $c' \in C'$, then $\ell' k' c' = 0$, and since δ is surjective, there exists $d \in D$ such that $\delta d = k'c'$. Using commutativity, we have that

$$0 = \ell' k' c' = \ell' \delta d = \varepsilon \ell d.$$

Since ε is injective, $\ell d = 0$. By exactness, there exists $c \in c$ such that $kc = d$. We get that

$$k'c' = \delta d = \delta kc = k' \gamma c.$$

Now consider $c' - \gamma c$, which satisfies $k'(c' - \gamma c) = 0$, and so by exactness there exists $b' \in B'$ such that $j'b' = c' - \gamma c$. Since β is surjective, there exists $b \in B$ such that $\beta b = b'$, and we get

$$c' - \gamma c = j' \beta b = \gamma j b.$$

Thus

$$c' = \gamma (c + jb)$$

and γ is surjective.

Now suppose $\gamma(c) = 0$. Then

$$0 = k' \gamma c = \delta k c.$$

since δ is injective, we have that $kc = 0$. Thus by exactness there exists $b \in B$ such that $jb = c$. Furthermore,

$$0 = \gamma c = \gamma j b = j' \beta b.$$

It follows from exactness that there is $a' \in A$ such that $i'a' = \beta b$. Since α is surjective, there exists $a \in A$ such that $\alpha a = a'$. It follows that

$$\beta b = i'a' = i' \alpha a = \beta i a.$$

Since β is injective, it follows that $b = ia$. Furthermore, it follows that $c = jb = jia = 0$. ☐

Proof of Theorem 41. First note that for any X and $A \subseteq X$, we have that $\Delta_n(X, A)$ includes into $C_n(X, A)$, giving chain maps. Let X^k be the k-skeleton, i.e., the set of simplices of dimension k or less. Then we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} H^{\Delta}_{n+1}(X^k, X^{k-1}) & \xrightarrow{\partial_*} & H^{\Delta}_n(X^{k-1}) & \xrightarrow{i_*} & H^{\Delta}_n(X^k) & \xrightarrow{q_*} & H^{\Delta}_n(X^k, X^{k-1}) & \xrightarrow{\partial_*} & H^{\Delta}_{n-1}(X^{k-1}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{n+1}(X^k, X^{k-1}) & \xrightarrow{\partial_*} & H_n(X^{k-1}) & \xrightarrow{i_*} & H_n(X^k) & \xrightarrow{q_*} & H_n(X^k, X^{k-1}) & \xrightarrow{\partial_*} & H_{n-1}(X^{k-1}) \end{array}$$

We will do an induction on k, assuming that second and fifth vertical maps is are isomorphisms for all n.

Furthermore, we show that first and fourth vertical maps are isomorphisms. For simplicial, it is clear that $\Delta_n\left(X^k, X^{k-1}\right) = \Delta_n\left(X^k\right)/\Delta_n\left(X^{k-1}\right)$ is zero if $n \neq k$ and is free abelian with basis the k-simplices if $n = k$. It follows that $H_n^\Delta\left(X^k, X^{k-1}\right)$ is the same. For simplicial, we see by definition of Δ-complex that $X^k/X^{k-1} \approx \coprod_\alpha \Delta_\alpha^k \,/\coprod_\alpha \partial\Delta_\alpha^k$. We can look at the exact sequence of the quotient:

$$\cdots \longrightarrow \tilde{H}_n\left(\coprod_\alpha \partial\Delta_\alpha^k\right) \xrightarrow{\iota_*} \tilde{H}_n\left(\coprod_\alpha \Delta_\alpha^k\right) \xrightarrow{j_*} \tilde{H}_n\left(\coprod_\alpha \Delta_\alpha^k \,/ \coprod_\alpha \partial\Delta_\alpha^k\right) \xrightarrow{\partial_*} \tilde{H}_{n-1}\left(\coprod_\alpha \partial\Delta_\alpha^k\right) \longrightarrow \cdots$$

All of the groups in the sequence are zero except $\tilde{H}_n\left(\coprod_\alpha \partial\Delta_\alpha^k\right)$ when $n = k-1$ or 0 and $\tilde{H}_n\left(\coprod_\alpha \Delta_\alpha^k\right)$ when $n = 0$. It follows that $\tilde{H}_n\left(\coprod_\alpha \Delta_\alpha^k \,/\coprod_\alpha \partial\Delta_\alpha^k\right)$ is a free abelian group with generators corresponding to the k-simplices. Since the maps we are interested in take the k-simplices in $H_k^\Delta\left(X^k, X^{k-1}\right)$ to the k-simplices in $H_k\left(X^k, X^{k-1}\right)$, these maps are isomorphisms. The theorem follows from the Five-Lemma. □

10. Euler characteristic

Definition 43. *The Euler characteristic of a Δ-complex X is*

$$\chi(X) = \sum_n (-1)^n c_n$$

where c_n is the number of cells of dimension n.

Proposition 44. *The Euler characteristic can be expressed*

$$\chi(X) = \sum_n (-1)^n \beta_n$$

where $\beta_n = rank\left(H_n(X)\right)$ are the Betti numbers.

Corollary 45. *The Euler characteristic is a topological invariant of the space.*

Proof. We will leave this as a homework exercise. □

11. Mayer-Vietoris sequence

Theorem 46 (Mayer-Vietoris)**.** *Let X be a topological space and A, B be subspaces such that the union of their interiors is X. Then there is a long exact sequence:*

$$\cdots \longrightarrow H_n(A\cap B) \xrightarrow{i_*\oplus j_*} H_n(A) \oplus H_n(B) \xrightarrow{k_*-\ell_*} H_n(X) \xrightarrow{\partial_*} H_{n-1}(A\cap B) \longrightarrow \cdots$$

where the maps are induced from inclusions

$$\begin{aligned} i &: A\cap B \to A \\ j &: A\cap B \to A \\ k &: A \to X \\ \ell &: B \to X. \end{aligned}$$

There is a similar one for reduced homology.

Example 3. *We can use this to compute the reduced homology of spheres again. where $S^n = D_1^n \cup D_2^n$ and $D_1^n \cap D_2^n \simeq S^{n-1}$. We get*

$$\cdots \longrightarrow \tilde{H}_n(D_1^n \cap D_2^n) \xrightarrow{i_*\oplus j_*} \tilde{H}_n(D_1^n) \oplus \tilde{H}_n(D_2^n) \xrightarrow{k_*-\ell_*} \tilde{H}_n(S^n) \xrightarrow{\partial_*} \tilde{H}_{n-1}((D_1^n \cap D_2^n)) \longrightarrow \cdots$$

and since $\tilde{H}_n(D_1^n) \oplus \tilde{H}_n\left(D_n^2\right) \cong 0$ we can perform induction.

Remark 8. *Much like Van Kampen, Mayer-Vietoris can be used to calculate homology groups of many spaces by decomposing the spaces into constituent parts.*

Example 4. *We can compute the homology groups of surfaces Σ_g of genus g inductively. Let the induction be the following:*

$$\tilde{H}_n(\Sigma_g) \cong \left\{ \begin{array}{c} \mathbb{Z} \text{ if } n=2 \\ \mathbb{Z}^{2g} \text{ if } n=1 \\ 0 \text{ otherwise} \end{array} \right\}$$

$$\tilde{H}_n(\Sigma_g \setminus D^2) \cong \left\{ \begin{array}{c} \mathbb{Z}^{2g} \text{ if } n=1 \\ 0 \text{ otherwise} \end{array} \right\}$$

where D^2 is a small disk and $g \geq 1$, and that the highest homology is generated by simplices covering Σ_g. We can calculate the top directly if $g=1$ using simplicial homology. We now consider the following two Mayer-Vietoris sequences:

$$\cdots \longrightarrow \tilde{H}_2(\partial D^2) \xrightarrow{i_* \oplus j_*} \tilde{H}_2(\Sigma_g \setminus D^2) \oplus \tilde{H}_2(D^2) \xrightarrow{k_* - \ell_*} \tilde{H}_2(\Sigma_g) \xrightarrow{\partial_*} \tilde{H}_1(\partial D^2) \longrightarrow \cdots$$

$$\cdots \xrightarrow{i_* \oplus j_*} \tilde{H}_1(\Sigma_g \setminus D^2) \oplus \tilde{H}_1(D^2) \xrightarrow{k_* - \ell_*} \tilde{H}_1(\Sigma_g) \xrightarrow{\partial_*} \tilde{H}_0(\partial D^2) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \tilde{H}_2(\partial D^2) \xrightarrow{i_* \oplus j_*} \tilde{H}_2(\Sigma_g \setminus D^2) \oplus \tilde{H}_2(\Sigma_1 \setminus D^2) \xrightarrow{k_* - \ell_*} \tilde{H}_2(\Sigma_{g+1}) \xrightarrow{\partial_*} \tilde{H}_1(\partial D^2) \longrightarrow \cdots$$

$$\cdots \xrightarrow{i_* \oplus j_*} \tilde{H}_1(\Sigma_g \setminus D^2) \oplus \tilde{H}_1(\Sigma_1 \setminus D^2) \xrightarrow{k_* - \ell_*} \tilde{H}_1(\Sigma_{g+1}) \xrightarrow{\partial_*} \tilde{H}_0(\partial D^2) \longrightarrow \cdots$$

The first we get

$$\cdots \longrightarrow 0 \xrightarrow{i_* \oplus j_*} \tilde{H}_2(\Sigma_g \setminus D^2) \xrightarrow{k_*} \tilde{H}_2(\Sigma_g) \xrightarrow{\partial_*} \tilde{H}_1(\partial D^2) \longrightarrow \cdots$$

$$\cdots \xrightarrow{i_* \oplus j_*} \tilde{H}_1(\Sigma_g \setminus D^2) \xrightarrow{k_* - \ell_*} \tilde{H}_1(\Sigma_g) \xrightarrow{\partial_*} 0$$

We see that the first ∂_* is an isomorphism since we know the generator of $\tilde{H}_2(\Sigma_g)$ and $\tilde{H}_1(\partial D^2) \cong \tilde{H}_1(S^1)$. It follows from exactness that the next map is the zero map, and the result for $\tilde{H}_n(\Sigma_g \setminus D^2)$ follows from the homology $\tilde{H}_n(\Sigma_g)$. In the second, we get

$$0 \xrightarrow{k_* - \ell_*} \tilde{H}_2(\Sigma_{g+1}) \xrightarrow{\partial_*} \tilde{H}_1(\partial D^2) \xrightarrow{i_* \oplus j_*} \tilde{H}_1(\Sigma_g \setminus D^2) \oplus \tilde{H}_1(\Sigma_1 \setminus D^2) \xrightarrow{k_* - \ell_*} \tilde{H}_1(\Sigma_{g+1}) \xrightarrow{\partial_*} 0$$

Since we know that i_* and j_* are the zero maps, induction follows.

To prove Mayer-Vietoris, we need only form the appropriate short exact sequence of chain complexes. The appropriate one is this:

$$0 \to C_n(A \cap B) \to C_n(A) \oplus C_n(B) \to C_n^{\{A,B\}}(A \cup B) \to 0,$$

where we know that the homology of the last chain complex gives the same homology as $C_n(A \cup B)$. Certainly $i_\# \oplus j_\#$ is injective and $k_\# - \ell_\#$ is surjective, so we need only show that

$$\ker(k_\# - \ell_\#) = \operatorname{Im}(i_\# \oplus j_\#).$$

If $\alpha \in C_n(A)$ and $\beta \in C_n(B)$ such that $\alpha - \beta = 0$ in $C_n^{\{A,B\}}(A \cup B)$, then clearly α and β have images only in $A \cap B$ and those come from the same simplex. The fact that $\operatorname{Im}(i_\# \oplus j_\#) \subseteq \ker(k_\# - \ell_\#)$ is clear. It follows that there is a long exact sequence.

For reduced Mayer-Vietoris, we see that the last line will be

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0,$$

and it is clear what the appropriate maps should be.

Note that the boundary maps ∂_* has a natural meaning. Suppose $[\xi] \in H_n(X)$. Then by subdivision we can represent $\xi = \alpha + \beta \in C_n(A) \oplus C_n(B)$ such that $\partial\alpha + \partial\beta = 0$ (since ξ is a cycle in X). It follows that $\partial\alpha = -\partial\beta \in C_{n-1}(A \cap B)$, and must be a cycle since $\partial^2 = 0$, so the map takes $[\xi]$ to $[\partial\alpha]$.

More examples: Compute for Klein bottle, real projective plane.

12. Degree

A map $f : S^n \to S^n$ induces a map $f_* : H_n(S^n) \to H_n(S^n)$. Since $H_n(S^n) \approx \mathbb{Z}$, the map f_* is multiplication by an integer. We call that number the degree. The degree is a homotopy invariant for maps $S^n \to S^n$. The degree can sometimes be computed fairly easily, and its definition preceded the definition of homology.

- $\deg Id = 1$.
- The degree of a reflection of $S^n \subseteq \mathbb{R}^{n+1}$ is -1.
- Compositions multiply degree: $\deg f \circ g = \deg f \deg g$. This follows from $(fg)_* = f_* g_*$.
- The antipodal map $S^n \to S^n$ given by $x \to -x$ has degree $(-1)^{n+1}$ since it consists of $n+1$ reflections (each changing the sign of one coordinate).
- A map $f : S^n \to S^n$ that is not surjective must have degree zero. This is because we can use stereographic projection to turn it into a map $\tilde{f} : S^n \to \mathbb{R}^n$, and this map is homotopic to a constant map. (The precise argument is this: Let $\phi : S^n \setminus \{pt\} \to \mathbb{R}^n$ be stereographic projection and $\phi^{-1} : \mathbb{R}^n \to S^n \setminus \{pt\}$ be its inverse. Then f is homotopic to a constant map via the homotopy $f_t(x) = \phi^{-1}(t(\phi(f(x))))$.)
- You can use degree theory to show some maps are not homotopic. For instance, the maps $S^1 \to S^1$ given by $z \to z^n$ are not homotopic to each other, since their degrees are n.

Many topological theorems can be proven using only degrees.

Theorem 47. *If n is even, then the only nontrivial group that can act freely (without fixed points) on S^n is $\mathbb{Z}_2$.*

Proof. Given a group acting on S^n, each element has a degree, and so there is a map $G \to \mathbb{Z}^\times$ (the multiplicative group of nonzero integers). The degree of a homeomorphism must be ± 1, and so the image of the map is $\mathbb{Z}_2$. □

Theorem 48. *S^n has a continuous, nonzero vector field if and only if n is odd.*

Proof. Suppose there is a nonzero vector field $v(x)$. We may replace $v(x)$ with $v(x) / |v(x)|$ and assume it has norm 1. That the vector field is tangent is described by the property $v(x) \cdot x = 0$, and so we note that for each x, $(\cos t) v(x) + (\sin t) x$ describes a circle in S^n. So letting t go between 0 and π, we have a homotopy between the identity and the antipodal map. Thus the degrees are the same, and so $(-1)^{n+1} = 1$, so n is odd.

The converse follows by considering the vector field $v(x_1, \ldots, x_{2k}) = (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1})$ on S^{2k-1}. □

Remark 9. *When $n = 2$, this is referred to as the hairy ball theorem.*

Remark 10. *Degrees can be computed locally, as follows. Suppose $y \in S^n$ and $f^{-1}(y) = \{x_1, \ldots, x_k\} \subseteq S^n$ (we are assuming the preimage is a finite number of* *points). Furthermore, let V be an open neighborhood of y and $U_1, \ldots, U_k$ be open neighborhoods of $x_1, \ldots, x_k$ that map into V. Then we can compute the local degree* $\deg f|_{x_i}$ *as the multiplier in the homomorphism*

$$f_* : H_n (U, U \setminus \{x_i\}) \to H_n (V, V \setminus \{y\}) .$$

Then the degree of f is the sum of the local degrees. If U_i are mapped homeomorphically to V, then local degrees are all ± 1, and you get the degree of the map by summing these local degrees.

13. First homology group

Proposition 49. *If X is path connected, the first homology group $H_1(X)$ is isomorphic to the abelianization of the fundamental group $\pi_1(X)$.*

Proof. Clearly there is a map $\pi_1(X) \to H_1(X)$ since every element of $\pi_1(X)$ is represented by a cycle. We need to compute the kernel of this map. Certainly the commutator subgroup is the kernel, since $H_1(X)$ is abelian. One needs to show that this is all of the kernel and that the map is surjective. We will not prove this, but it is in the book. □