VAN KAMPEN'S THEOREM

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1. Statement of theorem

Basic theorem:

Theorem 1. *If $X = A \cup B$, where A, B, and $A \cap B$ are path connected open sets each containing the basepoint $x_0 \in X$, then the inclusions*

$$j_A : A \to X$$
$$j_B : B \to X$$

induce a map

$$\Phi : \pi_1 (A, x_0) * \pi_1 (B, x_0) \to \pi_1 (X, x_0)$$

that is surjective. The kernel of Φ is the normal subgroup N generated by all elements of the form

$$\iota_{AB} (\omega) \iota_{BA} (\omega)^{-1}, \ \iota_{BA} (\omega) \iota_{AB} (\omega)^{-1}$$

where ι_{AB} and ι_{BA} are the homomorphisms

$$\iota_{AB} : \pi_1 (A \cap B, x_0) \to \pi_1 (A, x_0)$$
$$\iota_{BA} : \pi_1 (A \cap B, x_0) \to \pi_1 (B, x_0).$$

It follows that

$$\pi_1 (X, x_0) \cong \pi_1 (A, x_0) * \pi_1 (B, x_0) / N.$$

Remark 1. *Sometimes this is written*

$$\pi_1 (X, x_0) \cong \pi_1 (A, x_0) *_{\pi_2(A \cap B, x_0)} \pi_1 (B, x_0).$$

Full theorem:

Theorem 2. *If $X = \bigcup_\alpha A_\alpha$, where A_α and $A_\alpha \cap A_\beta$ are path connected open sets each containing the basepoint $x_0 \in X$, then the inclusions*

$$j_{A_\alpha} : A_\alpha \to X$$

induce a map

$$\Phi : *_\alpha \pi_1 (A_\alpha, x_0) \to \pi_1 (X, x_0)$$

that is surjective. If also each triple intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form

$$\iota_{\alpha\beta} (\omega) \iota_{\beta\alpha} (\omega)^{-1},$$

where $\iota_{\alpha\beta}$ are the homomorphisms

$$\iota_{\alpha\beta} : \pi_1 (A_\alpha \cap A_\beta, x_0) \to \pi_1 (A_\alpha, x_0)$$

It follows that $\pi_1 (X, x_0) \cong *_\alpha \pi_1 (A_\alpha, x_0) / N$.

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2. Free product

In the description of the theorem, we use the free product of groups.

Definition 3. *If* $\{G_\alpha\}_\alpha$ *are groups, the free product* $*_\alpha G_\alpha$ *consists of all words* $g_{\alpha_1} g_{\alpha_2} \cdots g_{\alpha_k}$ *(note that* k *is finite) that are reduced, i.e.,*

1. $g_{\alpha_i} \in A_{\alpha_i}$, $g_{\alpha_i} \neq e_{\alpha_i}$
2. G_{α_i} *and* $G_{\alpha_{i+1}}$ *are different groups .*

This forms a group by juxtaposition and reduction. (The empty word is the identity.)

Remark 2. *The difficult part of showing this is a group is that the multiplication is associative. See the book for details.*

Example 1. $\mathbb{Z} * \mathbb{Z}$ *is given by two generators* a, b*. Typical elements are*

$$a^2 b^{-1} a^{-4},\ b,\ b^{10} a^2.$$

Multiplication can be done, for example:

$$\left(aba^{-1}\right)\left(ab^{-1}a^{10}\right) = a^{11}.$$

The key fact about free products is the following.

Proposition 4. *Any collection of homomorphisms* $h_\alpha : G_\alpha \to G$ *induce a unique homomorphism*

$$h : *_\alpha G_\alpha \to G$$

by $h\left(g_{\alpha_1} \cdots g_{\alpha_k}\right) = h_{\alpha_1}\left(g_{\alpha_1}\right) \cdots h_{\alpha_k}\left(g_{\alpha_k}\right).$

Proof. Since h_{α_i} are homomorphisms, this is a well-defined homomorphism. □

3. Examples

- Consider two loops joined together at a point x_0. This space is called $S^1 \vee S^1$ and is called the wedge product of two circles. Each S^1 has a generator, and then the intersection is contractible, so we get $\pi_1\left(S^1 \vee S^1, x_0\right) \cong \mathbb{Z} * \mathbb{Z}$.
- In general, any wedge sum, $\bigvee_\alpha X_\alpha$ of spaces, defined as the quotient where one point is identified in each space is the free product of the fundamental groups if each basepoint is a deformation retraction of a neighborhood.
- We can look at the complement of a circle in $\mathbb{R}^3$. It is not too hard to see that $S^1 \vee S^2$ is a deformation retract of this space, so the fundamental group is $\mathbb{Z}$. The complement of two unlinked circles can be deformation retracted to $S^1 \vee S^1 \vee S^2 \vee S^2$, so the fundamental group is $\mathbb{Z} * \mathbb{Z}$. The complement of two linked circles can be deformation retracted to $\left(S^1 \times S^1\right) \vee S^2$.
- Another proof that the torus is $\mathbb{Z} \times \mathbb{Z}$. take the complement of a point as one open set, and a small disk as the other (with the the basepoint in the intersection). One is contractible, the other can be contracted to $S^1 \vee S^1$, and then the intersection can be deformation retracted to S^1. You will find that the inclusions induce a quotient by the commutator subgroup.
- You can also look at the sphere. One can make the sphere S^n a union of two disks homeomorphic to D^n, and the intersection is homeomorphic to $(0,1) \times S^{n-1}$. Since the two disks have trivial fundamental group, so does the free product. Note that this does not work for S^1, since the intersection is not path connected!

- Another thing we can see is that the complement of a circle S^1 in S^3 has the same fundamental group as the complement in $\mathbb{R}^3$, since we simply start with $S^3 \setminus S^1$, then take a point out, and use Van Kampen. Since the intersection is simply connected and a disk is simply connected, we get that the two fundamental groups are isomorphic.
- We can consider the Klein bottle in two ways: as the union of a disk and its complement, leading to the group
$$\mathbb{Z} * \mathbb{Z}/xyx^{-1}y = 1$$
and the union of two mobius strips, leading to the group
$$\mathbb{Z} * \mathbb{Z}/a^2b^2 = 1.$$
Notice the map generated by $a \to xy^{-1}$ and $b \to yx^{-1}y$ is an isomorphism since
$$a^2b^2 \to xy^{-1}xy^{-1}yx^{-1}yyx^{-1}y = xyx^{-1}y$$
and its inverse is generated by $x \to a^2b$ and $y \to ab$ since
$$xyx^{-1}y \to a^2babb^{-1}a^{-2}ab = a^2b^2.$$
- Why do we need triple intersections to be path connected in the general version? Consider the union of an annulus and two disks such that the intersection of the disks goes through the hole and so the triple intersection is not path connected, but the union of all disks is contractible. The free product gives $\mathbb{Z}$, but each intersection is contractible, so we would get nothing in the kernel if the kernel was as described, clearly false.

4. Proof

The idea of the proof is that given any loop in X, it can be partitioned into pieces in A or B, which can be made into loops by connecting back to the basepoint (since the intersection is path connected, and contains the basepoint). The only way a loop in A can be trivial is if there is a homotopy to the intersection, and then a homotopy to constant within B. This is essentially the condition given.

Proof of Theorem 1. Let γ be a loop based at x_0. Each point $s \in [0,1]$ is contained in an open interval I_s such that $\gamma(I_s)$ is contained in A or B. These form a cover of I, and so there is a finite subcover, giving a partition $0 = s_0 < s_1 < \cdots < s_k = 1$ such that $\gamma([s_i, s_{i+1}])$ is contained in A or B. We can extend these so that $\gamma(s_i) \in A \cap B$ by combining intervals that do not satisfy this, and relabeling. For each i, since $A \cap B$ is connected, there is a path α_i from x_0 to $\gamma(s_i)$. Let $\gamma_i = \gamma|_{[s_i, s_{i+1}]}$. We can now decompose γ as
$$\begin{aligned}
[\gamma]_X &= [\gamma_0 \cdot \cdots \cdot \gamma_{k-1}]_X \\
&= [\gamma_0 \cdot \bar{\alpha}_1 \cdot \alpha_1 \cdot \cdots \cdot \bar{\alpha}_{k-1} \cdot \alpha_{k-1} \cdot \gamma_{k-1}]_X \\
&= [\gamma_0 \cdot \bar{\alpha}_1]_X [\alpha_1 \cdot \gamma_1 \cdot \bar{\alpha}_2]_X \cdots [\alpha_{k-1} \cdot \gamma_{k-1}]_X \\
&= j_*([\gamma_0 \cdot \bar{\alpha}_1]) j_*([\alpha_1 \cdot \gamma_1 \cdot \bar{\alpha}_2]) \cdots j_*([\alpha_{k-1} \cdot \gamma_{k-1}])
\end{aligned}$$
where j is the inclusion of A or B into X. This is clearly in the image of Φ. This proves surjectivity.

Next, we want to make sure that the quotient map
$$\bar{\Phi} : \pi_1(A, x_0) * \pi_1(B, x_0)/N \to \pi_1(X, x_0)$$

is well-defined. We see that $\Phi\left(\iota_{AB}(\omega)\,\iota_{BA}^{-1}(\omega)\right) = [\omega\cdot\bar{\omega}]$, which is contractible, so it follows that the map on the quotient is well defined. To show the second part, we will show that $\bar{\Phi}$ is injective. We first suppose that $\Phi\left([f_1]_{U_1}[f_2]_{U_2}\cdots[f_k]_{U_k}\right) = [f_1\cdot f_2\cdot\cdots\cdot f_k]_X = e$. Thus there is a homopy $H : I\times I\to X$ between $f_1\cdot f_2\cdot\cdots\cdot f_k$ and the constant loop.

Claim 1. *There is a partition* $0 = s_1 < s_2 < \cdots < s_m = 1$ *and* $0 = t_1 < \cdots < t_n = 1$ *such that* $H([s_i, s_{i+1}]\times[t_j, t_{j+1}])$ *is contained in either A or B, which we will denote as* U_{ij}.

To prove the claim, we note that each point $(s,t)\in I\times I$ is contained in the pre-image of one of A or B, and open set in the product topology. Thus there is a rectangle whose closure maps into one of these two sets for each point (s,t), then by compactness there is a finite subcover. Now take all possible s and t coordinates of these rectangles and take this as the partition.

If the image of the corners of the rectangles lie in $A\cap B$, there is a path from x_0 to the image of the corner that stays entirely within $A\cap B$ (since this is path connected), and so we can alter the homotopy such that the corners all map to x_0 (if the corner is not in $A\cap B$, then all the rectangles meeting at the point map to the same of A or B, and so we can take a path with in that set). We may also assume that each of $f_1, f_2, \ldots, f_k$ are loops in $U_1 = U_{11}$ and whose endpoints come from corners of the rectangles.

We now move through the homotopy one rectangle at a time, as this corresponds to a homotopy of loops in U_{ij}. The key is that the loops at the boundary of rectangles mapping to different sets are in the intersection, and so the quotient allows us to rewrite free products as follows: (let $f_1,\ldots$ be the top horizontal row, $g_1,\ldots$ be the bottom horizontal row for one rectangle, and $h_0, h_1,\ldots$ be the vertical row pointing down)

$$\begin{aligned}[f_1]_{U_1}[f_2]_{U_2} &= \left[h_0\cdot g_1\cdot\bar{h}_1\right]_{U_1}[h_1\cdot g_2\cdot h_2]_{U_2}\\ &= [h_0\cdot g_1]_{U_1}[g_2\cdot h_2]_{U_2}\end{aligned}$$

where the second equality is due to the quotient. Since the end paths are constant, we get the appropriate equality between the top and bottom. this completes the proof. □

Remark 3. *The general case is not that different. You just need to perturb the rectangles so only three meet at any corner. Then if you take the path to the endpoint to be in the triple intersection, you can do the same procedure.*