Math 537B Homework 1

January 17, 2006

1) 7-1

2) 7-2

3) Recall that the Heisenberg group N consists of 3×3 unit upper triangular matrices, and that if given the coordinates

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

its Lie algebra has a basis of left invariant vector fields

$$F_1 = \frac{\partial}{\partial x}$$
$$F_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$$
$$F_3 = \frac{\partial}{\partial z}$$

(convince yourselves that these are, in fact, left invariant, i.e. $(L_\gamma)_* F_i = F_i$ for any $\gamma \in N$, where $L_\gamma : N \to N$, $L_\gamma(\alpha) = \gamma\alpha$ is multiplication in the group on the left). Note that

$$[F_1, F_2] = F_3$$
$$[F_2, F_3] = [F_3, F_1] = 0.$$

We may introduce a left-invariant Riemannian metric on N by declaring that $g(F_i, F_j) = A_i \delta_{ij}$ (so $g(F_i, F_i) = A_i$ and $g(F_i, F_j) = 0$ if $i \neq j$). Note that $F_i g(F_j, F_k) = 0$ for all i, j, k (since $g(F_j, F_k)$ is constant). This space is often called Nil (since the Lie algebra is nilpotent) and its geometry is called nilgeometry.

a) Compute the connection by computing $\nabla_{F_i} F_j$ for all i, j. (Hint: use the last observation to simplify your calculation.)

b) Compute the Riemannian curvature tensor for Nil, i.e. $R(F_i, F_j, F_k, F_\ell)$.

c) Show that all compact quotients of Nil are almost flat. A manifold M is almost flat if for any $\varepsilon > 0$ there exists a Riemannian metric on M such that

$$|K| D^2 \leq \varepsilon$$

for all sectional curvatures K, where D is the diameter,

$$D = \sup\{d(p, q) : p, q \in \text{Nil}\}.$$

(Note that $|K| D^2$ is scale-invariant, i.e. it does not change if the metric g is replaced by cg for any constant $c > 0$.) Hint: Making A_i smaller can only reduce the diameter.