Math 537B Homework 4 (Optional)

April 24, 2006

1) Recall that in Homework 3 we derived that the metric in normal coordinates can be expanded as a Taylor series as

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikj\ell}(0)\, x^k x^\ell + O\left(|x|^3\right).$$

Use this and the well-known formulas

$$\frac{d}{dt} \log \det g = g^{ij} \frac{d}{dt} g_{ij}$$

$$\frac{d}{dt} g^{ij} = -g^{ik} \left(\frac{d}{dt} g_{k\ell}\right) g^{\ell j}$$

to show that the Taylor series for $\det g$ is

$$\det g(x) = 1 - \frac{1}{3} R_{ij}(0)\, x^i x^j + O\left(|x|^3\right)$$

and that the Taylor series for the volume of the ball of radius r, called $V(r)$, is

$$V(r) = \left(1 - \frac{1}{6(n+2)} S(0)\, r^2 + O\left(r^3\right)\right) \frac{r^n \omega_{n-1}}{n}$$

where ω_{n-1} is the $(n-1)$-dimensional volume of the sphere of radius 1 in $\mathbb{R}^n$ and S is the scalar curvature. Note that $r^n \omega_{n-1}/n$ is equal to the volume of the Euclidean ball of radius r.

2) Recall that the Lie derivative of a covariant tensor T is defined as

$$\mathcal{L}_X T = \lim_{t \to 0} \frac{\phi_t^* T - T}{t}$$

where ϕ_t is a 1-parameter family of diffeomorphisms $\phi_t : M \to M$ generated by the vector field X. Show that if g_{ij} is the Riemannian metric tensor, then

$$\left(\mathcal{L}_X g\right)_{ij} = \nabla_i X_j + \nabla_j X_i$$

where ∇ is the Riemannian connection and $X_j = g_{jk} X^k$ if $X = X^k \frac{\partial}{\partial x^k}$. Conclude that if $g_{ij}(t) = \phi_t^* g_{ij}(0)$ is a solution to the normalized Ricci flow, then

$$(r - R)\, g_{ij} = \nabla_i X_j + \nabla_j X_i.$$

A Riemannian metric satisfying this equation (in two dimensions) for some vector field X is called a *Ricci-soliton*. The Ricci soliton equation in higher dimensions is

$$r g_{ij} - R_{ij} = \nabla_i X_j + \nabla_j X_i.$$