Global Differential Geometry HW 1

Due September 6, 2011

1) Consider the helix given by $t \to (\cos t, \sin t, t)$.
a) Reparametrize the helix by arclength.
b) Compute the curvature and torsion.
c) Show that the energy

$$E(\gamma) = \int_a^b |\dot{\gamma}|^2 \, dt$$

of any finite piece of the curve is not independent of reparametrization.

2) Compute its first and second fundamental forms, as well as mean and Gaussian curvatures of the following surfaces:
a) the paraboloid $z = x^2 + y^2$
b) the saddle $z = x^2 - y^2$
c) the trough $z = x^2$

3) In this problem we review different definitions of the tangent bundle. Let M^n be a smooth n-dimensional manifold with smooth atlas $\{U_i, \phi_i\}_{i \in I}$ where $U_i \subset M$ is open and $\phi_i : U_i \to \mathbb{R}^n$ is an embedding (diffeomorphic onto its image). Consider the following three definitions:

Definition 1 *$T^{glue}M = \bigsqcup_i (\phi_i(U_i) \times \mathbb{R}^n) / \sim$ where for $(x, v) \in \phi_i(U_i) \times \mathbb{R}^n$ and $(y, w) \in \phi_i(U_i) \times \mathbb{R}^n$ we have $(x, v) \sim (y, w)$ if and only iff $y = \phi_j \phi_i^{-1}(x)$ and $w = d\left(\phi_j \phi_i^{-1}\right)_x (v)$ ($\bigsqcup$ stands for disjoint union). We also define the fiber as $T_p^{glue}M = \bigsqcup_{\phi_i^{-1}(x) = p} (\{x_i\} \times \mathbb{R}^n) / \sim$. Notice that $T_p^{glue}M \subset T^{glue}M$.*

Definition 2 *$T_p^{path}M = \{paths\ \gamma : (-\varepsilon, \varepsilon) \to M \text{ such that } \gamma(0) = p\} / \sim$ where $\alpha \sim \beta$ if $(\phi_i \circ \alpha)'(0) = (\phi_i \circ \beta)'(0)$ for every i such that $p \in U_i$. Also define $T^{path}M = \bigsqcup_{p \in M} T_p^{path}M$.*

Definition 3 *$Germs_p$ is the set of functions $f \in C^\infty(U_f)$ for $p \in U_f \subset M$ modulo the equivalence that $[f] = [g]$ iff $f(x) = g(x)$ for all $x \in U_f \cap U_g$. Note that $Germs_p$ are a vector space since $[f] + [g] = [f + g]$ is well-defined, etc.*

Definition 4 *A derivation of germs is an* $\mathbb{R}$*-linear map* $X: Germs_p \to \mathbb{R}$ *which satisfies*

$$X(fg) = f(p)X(g) + X(f)g(p)$$

for any $f, g \in Germs_p$.

Definition 5 *We define* $T_p^{der}M$ *to be the set of derivations of germs at* p*. Also define* $T^{der}M = \bigsqcup_{p \in M} T_p^{der}M$

a) Show that $T^{\text{glue}}M \cong T^{\text{path}}M \cong T^{\text{der}}M$ as bundles (i.e., there are smooth diffeomorphisms between them which preserve the fibers $T_p^{***}M$. For this reason, we will refer only the tangent bundle TM.

b) Explain what the basis $\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$ of T_pM (corresponding to local coordinates $\left(x^1, \ldots, x^n\right)$) represent in each of the definitions.