Global Differential Geometry HW 2

Due September 27, 2011

1) A Lie group G is a differential manifold that is also a group such that the maps $L_a : G \to G$ for any $a \in G$ and $Inv : G \to G$ given by

$$L_a(\gamma) = a\gamma$$
$$Inv(\gamma) = \gamma^{-1}$$

are smooth maps (and hence diffeomorphisms). The tangent space at the identity T_eG is called the Lie algebra and denoted as $\mathfrak{g}$. One can define a Riemannian metric on G by fixing an inner product g_e on $\mathfrak{g}$ and then for any vectors $X_\gamma, Y_\gamma \in T_\gamma G$, defining

$$g(X_\gamma, Y_\gamma) = g_e\left(L_{\gamma^{-1}*}X_\gamma, L_{\gamma^{-1}*}Y_\gamma\right)$$

(note that $L_{\gamma^{-1}*} : T_\gamma G \to T_eG = \mathfrak{g}$). Such a metric is called a left-invariant metric on G.

a) Show that the metric really is left-invariant, in the sense that for any $\gamma \in G$, $L_\gamma^* g = g$.

b) The Lie group Nil is the three-dimensional Lie group of unit upper triangular 3×3 matrices with the usual matrix multiplication. With the coordinates:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

compute the left-invariant metric such that $\left.\frac{\partial}{\partial x}\right|_e, \left.\frac{\partial}{\partial y}\right|_e, \left.\frac{\partial}{\partial z}\right|_e$ are orthonormal.

2) Lee, Exercise 3.8

3) Lee, Problem 3-3