Global Differential Geometry HW 1

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1) In this problem we review the theory of curves. It will be helpful later in the course. Recall the following definitions:

Definition 1 *A parametrized curve is a C^∞ map $\gamma : (a, b) \to \mathbb{R}^n$. A parametrized curve is regular if $\gamma'(t) \neq 0$ for all $t \in (a, b)$.*

Definition 2 *A parametrized curve $\beta : (c, d) \to \mathbb{R}^n$ is a reparametrization of γ if there exists an invertible C^∞ map $\rho : (c, d) \to (a, b)$ with $\rho' > 0$ such that $\beta = \gamma \circ \rho$.*

Definition 3 *The arclength of a regular parametized curve can be expressed by*

$$L(\gamma) = \int_a^b |\dot{\gamma}| \, dt.$$

Show that arclength is independent of reparametrization. Show that the energy

$$E(\gamma) = \int_a^b |\dot{\gamma}|^2 \, dt$$

is not independent of reparametrization. Show that any curve may be reparametrized by arclength, meaning that it may be reparametrized into $\gamma(s)$ with $a \in [0, L]$ such that $\int_0^{s_0} \left| \frac{d\gamma}{ds} \right| ds = s_0$. Show this is equivalent to $\left| \frac{d\gamma}{ds} \right| = 1$.

2) Recall a smooth manifold M is a manifold with charts $\{U_i, \phi_i\}$, where $\phi_i : U_i \to \mathbb{R}^n$ is an embedding such that overlap maps $\phi_i \circ \phi_j^{-1}$ are C^∞.

a) Recall that a surface is a map $X : U \to \mathbb{R}^3$ such that $U \subset \mathbb{R}^2$ is open and dX has full rank at every point in U. Explain why a surface is a manifold. (Hint: Use the inverse function theorem.)

b)Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Show that the level set $f(x^1, \ldots, x^n) = 0$ is a smooth manifold if df is rank 1 at every point in the level set. (Hint: Use the implicit function theorem.)

3) In this problem we review different definitions of the tangent bundle. Let M^n be a smooth n-dimensional manifold with smooth atlas $\{U_i, \phi_i\}_{i\in I}$ where $U_i \subset M$ is open and $\phi_i : U_i \to \mathbb{R}^n$ is an embedding (diffeomorphic onto its image). Consider the following three definitions:

Definition 4 $T^{glue}M = \bigsqcup_i (\phi_i(U_i) \times \mathbb{R}^n) / \sim$ *where for* $(x,v) \in \phi_i(U_i) \times \mathbb{R}^n$ *and* $(y,w) \in \phi_i(U_i) \times \mathbb{R}^n$ *we have* $(x,v) \sim (y,w)$ *if and only iff* $y = \phi_j\phi_i^{-1}(x)$ *and* $w = d\left(\phi_j\phi_i^{-1}\right)_x(v)$ *($\bigsqcup$ stands for disjoint union). We also define the fiber as* $T_p^{glue}M = \bigsqcup_{\phi_i^{-1}(x)=p} (\{x_i\} \times \mathbb{R}^n) / \sim$. *Notice that* $T_p^{glue}M \subset T^{glue}M$.

Definition 5 $T_p^{path}M = \{paths\ \gamma : (-\varepsilon, \varepsilon) \to M \text{ such that } \gamma(0) = p\} / \sim$ *where* $\alpha \sim \beta$ *if* $(\phi_i \circ \alpha)'(0) = (\phi_i \circ \beta)'(0)$ *for every* i *such that* $p \in U_i$. *Also define* $T^{path}M = \bigsqcup_{p\in M} T_p^{path}M$.

Definition 6 *$Germs_p$ is the set of functions* $f \in C^\infty(U_f)$ *for* $p \in U_f \subset M$ *modulo the equivalence that* $[f] = [g]$ *iff* $f(x) = g(x)$ *for all* $x \in U_f \cap U_g$. *Note that $Germs_p$ are a vector space since* $[f] + [g] = [f+g]$ *is well-defined, etc.*

Definition 7 *A derivation of germs is an $\mathbb{R}$-linear map* $X : Germs_p \to \mathbb{R}$ *which satisfies*

$$X(fg) = f(p)\, X(g) + X(f)\, g(p)$$

for any $f, g \in Germs_p$.

Definition 8 *We define* $T_p^{der}M$ *to be the set of derivations of germs at p. Also define* $T^{der}M = \bigsqcup_{p\in M} T_p^{der}M$

a) Show that $T^{\text{glue}}M \cong T^{\text{path}}M \cong T^{\text{der}}M$ as bundles (i.e., there are smooth diffeomorphisms between them which preserve the fibers $T_p^{***}M$. For this reason, we will refer only the tangent bundle TM.

b) Explain what the basis $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ of T_pM (corresponding to local coordinates $\left(x^1, \ldots, x^n\right)$) represent in each of the definitions.