Introduction to Poincare Conjecture and the Hamilton-Perelman program

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1 Introduction

This lecture is mostly taken from Tao's lecture 2.

In this lecture we are going to introduce the Poincaré conjecture and the idea of the proof. The rest of the class will be going into different details of this proof in varying amounts of careful detail. All manifolds will be assumed to be without boundary unless otherwise specified.

The Poincaré conjecture is this:

Theorem 1 (Poincare conjecture) *Let M be a compact 3-manifold which is connected and simply connected. Then M is homeomorphic to the 3-sphere $\mathbb{S}^3$.*

Remark 2 *A simply connected manifold is necessarily orientable.*

In fact, one can prove a stronger statement called Thurston's geometrization conjecture, which is quite a bit more complicated, but is roughly the following:

Theorem 3 (Thurston's geometrization conjecture) *Every 3-manifold M can be cut along spheres and π_1-essential tori such that each piece can be given one of 8 geometries ($\mathbb{E}^3, \mathbb{S}^3, \mathbb{H}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \widetilde{\mathrm{SL}}(2, \mathbb{R}), \mathrm{Nil}, \mathrm{Sol}$).*

We may go into this conjecture a little more if we have time, but certainly elements of this will come up in the process of these lectures. We will take a quick look

2 Examples of 3-manifolds

Here we look in the topological category. It turns out that in three dimensions, the smooth category, the piecewise linear category, and the topological category are the same (e.g., any homeomorphism can be approximated by a diffeomorphism, any topological manifold can be given a smooth structure, etc.)

2.1 Spherical manifolds

The most basic 3-manifold is the 3-sphere, S^3, which can be constructed in several ways, such as:

- The set of unit vectors in $\mathbb{R}^4$
- The one point compactification of $\mathbb{R}^3$.

Using the second definition, it is clear that any loop can be contracted to a point, and so S^3 is simply connected. One can also look at the first definition and see that the rotations $SO(4)$ of $\mathbb{R}^4$ act transitively on S^3, with stabilizer $SO(3)$, and so S^3 is a homogeneous space described by the quotient $SO(4)/SO(3)$. One can also notice that the unit vectors in $\mathbb{R}^4$ can be given the group structure of the unit quaternions, and it is not too hard to see that this group is isomorphic to $\mathrm{SU}(2)$, which is the double cover of $SO(3)$.

As more examples of 3-manifolds, it is possible to find finite groups Γ acting freely on S^3, and consider quotients of S^3 by the action. Note that if the action is nontrivial, then these new spaces are not simply connected. One can see this in several ways. The direct method is that since there must be $g \in \Gamma$ and $x \in S^3$ such that $gx \neq x$. A path γ from x to gx in S^3 descends to a loop in S^3/Γ. If there is a homotopy of that loop to a point, then one can lift the homotopy to a homotopy $H : [0,1] \times [0,1] \to S^3$ such that $H(t,0) = \gamma(t)$. But then, looking at $H(1,s)$, we see that $H(1,0) = gx$ and $H(1,1) = x$ and $H(1,s) = g'x$ for some $g' \in \Gamma$ for all s. But since Γ is discrete, this is impossible. A more high level approach shows that the map $S^3 \to S^3/\Gamma$ is a covering map, and, in fact, the universal covering map and so $\pi_1\left(S^3/\Gamma\right) = \Gamma$ if Γ acts effectively. Hence each of these spaces S^3/Γ are different manifolds than S^3. They are called spherical manifolds.

The elliptization conjecture states that spherical manifolds are the only manifolds with finite fundamental group.

2.2 Sphere bundles over S^1

The next example is to consider S^2 bundles over S^1, which is the same as $S^2 \times [0,1]$ with $S^2 \times \{0\}$ identified with $S^2 \times \{1\}$ by a homeomorphism. Recall that a homeomorphism

$$\phi : S^2 \to S^2$$

induces an isomorphism on homology (or cohomology),

$$\phi_* : H_2\left(S^2\right) \to H_2\left(S^2\right)$$

and since

$$H_2\left(S^2\right) \cong \mathbb{Z}$$

there are only two possibilities for ϕ_*. In fact, these classify the possible maps up to continuous deformation, and so there is an orientation preserving and an orientation reversing homeomorphism and that is all (using π_2 instead of H_2). (See, for instance, Bredon, Cor. 16.4 in Chapter 2.)

Notice that these manifolds have a map $\phi : M \to S^1$. It is clear that this induces a surjective homomorphism on fundamental group $\phi_* : \pi_1(M) \to \pi_1(S^1) \cong \mathbb{Z}$. The kernel of this map consists of loops which can be deformed to maps only on S^2 (constant on the other component), and since S^2 is simply connected, this map is an isomorphism. Note that these manifolds are thus not simply-connected.

2.3 Connected sum

One can also form new manifolds via the connected sum operation. Given two 3-manifolds, M and M', one forms the connected sum by removing a disk from each manifold and then identifying the boundary of the removed disks. We denote this as $M\#M'$. Recall that in 2D, all manifolds can be formed from the sphere and the torus in this way.

Now, we may consider the class of all compact, connected 3-manifolds (up to homeomorphism) with the connected sum operation. These form a monoid (essentially a group without inverse), with an identity (S^3). Any nontrivial (i.e., non-identity) manifold can be decomposed into pieces by connected sums, i.e., given any M, we can write

$$M \approx M_1\#M_2\#\cdots\#M_k$$

where M_j cannot be written as a connected sum any more (this is a theorem of Kneser). We call such a decomposition a prime decomposition and such manifolds M_j prime manifolds. The proof is very similar to the fundamental theorem of arithmetic which gives the prime decomposition of positive integers.

Proposition 4 *Suppose M and M' are connected manifolds of the same dimension. Then*

1. *$M\#M'$ is compact if and only if both M and M' are compact.*
2. *$M\#M'$ is orientable if and only if both M and M' are orientable.*
3. *$M\#M'$ is simply connected if and only if both M and M' are simply connected.*

We leave the proof as an exercise, but it is not too difficult.

3 Idea of the Hamilton-Perelman proof

In order to give the idea, we will introduce a few concepts which will be defined more precisely in successive lectures.

Any smooth manifold can be given a Riemannian metric, denoted g_{ij} or $g(\cdot,\cdot)$, which is essentially an inner product (i.e., symmetric, positive-definite bilinear form) at each tangent space which varies smoothly as the basepoint of the tangent space changes. The Riemannian metric allows one to define angles between two curves and also to measure lengths of (piecewise C^1-) curves by integrating the tangent vectors of a curve. That is, if $\gamma : [0, a] \to M$ is a curve, we can calculate its length as

$$\ell(\gamma) = \int_0^a g(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2} \, dt,$$

where $\dot{\gamma}(t)$ is the tangent to the curve at t. A Riemannian metric induces a metric space structure on M, as the distance between two points is given by the infimum of lengths of all piecewise smooth curves from one point to the other. It is a fact that the metric topology induces the original topology of the manifold.

The main idea is to deform any Riemannian metric to a standard one. This is the idea of "geometrizing." How does one choose the deformation? R. Hamilton first proposed to deform by an equation called the Ricci flow, which is a partial differential equation defined by

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} = -2 \operatorname{Rc}(g_{ij})$$

where t is an extra parameter (not related to the original coordinates, so $g_{ij} = g_{ij}(t, x)$, where x are the coordinates) and $R_{ij} = \operatorname{Rc}(g_{ij})$ is the Ricci curvature, a differential operator (2nd order) on the Riemannian metric. That means that the Ricci flow equation is a partial differential equation on the Riemannian metric. In coordinates, it roughly looks like

$$\frac{\partial}{\partial t} g_{ij} = -2g^{k\ell} \frac{\partial^2}{\partial x^k \partial x^\ell} g_{ij} + F(g_{ij}, \partial g_{ij})$$

where $g^{k\ell}$ is the inverse matrix of g_{ij} and F is a function depending only on the metric and first derivatives of the metric.

3.1 Hamilton's first result

The idea is that as the metric evolves, its curvature becomes more and more uniform. It was shown in Hamilton's landmark 1982 paper that

Theorem 5 (Hamilton) *Given a Riemannian 3-manifold (M, g_0) with positive Ricci curvature, then the Ricci flow with $g(0) = g_0$ exists on a maximal time interval $[0, t_*)$. Furthermore, the Ricci curvature of the metrics $g(t)$ become increasingly uniform as $t \to t_*$. More precisely,*

$$R_{ij}(t) - \bar{R}(t) g_{ij}(t),$$

where $\bar{R}$ is the average scalar curvature, converges uniformly to zero as $t \to t_0$.

From this, one can easily show that a rescaling of the metric converges to the round sphere.

3.2 2D case

In the 2D case it can be shown that any compact, orientable Riemannian manifold converges under a renormalized Ricci flow (renormalized by rescaling the metric and rescaling time) to a constant curvature metric. This is primarily due to Hamilton, with one case finished by B. Chow. It is possible to use this method to prove the uniformization theorem, which states that any compact, orientable Riemannian manifold can be conformally deformed to a metric with constant curvature. (The original proofs of Hamilton and Chow use the uniformization theorem, but a recent article by Chen, Lu, and Tian shows how to avoid that).

3.3 General case

Hamilton introduced a program to study all 3-manifolds using the Ricci flow. It was discovered quite early that the Ricci flow may develop singularities even in the case of a sphere if the Ricci curvature is not positive. An example is the so-called neck pinch singularity. Hamilton's idea was to do surgery at these singularities, then continue the flow and continue to do this until no more surgeries are necessary. Perelman's work describes what happens to the Ricci flow near a singularity and also how to perform the surgery. The new flow is called *Ricci Flow with surgery.* The main result of Perelman is the following.

Theorem 6 (Existence of Ricci flow with surgery) *Let (M, g) be a compact, orientable Riemannian 3-manifold. Then there exists a Ricci flow with surgery $t \to (M(t), g(t))$ for all $t \in [0, \infty)$ and a closed set $T \subset [0, \infty)$ of surgery times such that:*

1. *(Initial data) $M(0) = M$, $g(0) = g$.*
2. *(Ricci flow) If I is any connected component of $[0, \infty) \setminus T$ (and thus an interval), then $t \to (M(t), g(t))$ is the Ricci flow on I (you can close this interval on the left endpoint if you wish).*
3. *(Topological compatibility) If $t \in T$ and $\varepsilon > 0$ is sufficiently small, then we know the topological relationship $M(t - \varepsilon)$ and $M(t)$.*
4. *(Geometric compatibility) For each $t \in T$, the metric $g(t)$ on $M(t)$ is related to a certain limit of the metric $g(t - \varepsilon)$ on $M(t - \varepsilon)$ by a certain surgery procedure.*

Note, we can express the topological compatibility more precisely. We have that $M(t - \varepsilon)$ is homeomorphic to the connected sum of finitely many connected components of $M(t)$ together with a finite number of spherical space forms (spherical manifolds), $\mathbb{RP}^3 \# \mathbb{RP}^3$, and $S^2 \times S^1$. Furthermore, each connected component of $M(t)$ is used in the connected sum decomposition of exactly one component of $M(t - \varepsilon)$.

Remark 7 *The case of* $\mathbb{RP}^3\#\mathbb{RP}^3$ *is interesting in that it is apparently the only nonprime 3-manifold which admits a geometric structure (i.e., is covered by a model geometry; it is a quotient of* $S^2\times\mathbb{R}$*; this does not contradict our argument above because it is not a sphere bundle over* S^1*). I have seen this mentioned several places, but I do not have a reference.*

Remark 8 *Morgan-Tian and Tao give a more general situation where nonorientable manifolds are allowed. This adds some extra technicalities which we will avoid in this class.*

The existence needs something more to show the Poincaré conjecture. One needs that the surgeries are only discrete and that the flow shrinks everything in finite time.

Theorem 9 (Discrete surgery times) *Let* $t\rightarrow(M(t),g(t))$ *be a Ricci flow with surgery starting with an orientable manifold* $M(0)$*. Then the set* T *of surgery times is discrete. In particular, any compact time interval contains a finite number of surgeries.*

Theorem 10 (Finite time extinction) *Let* (M,g) *be a compact* 3*-manifold which is simply connected and let* $t\rightarrow(M(t),g(t))$ *be an associated Ricci flow with surgery. Then* $M(t)$ *is empty for sufficiently large* t*.*

With these theorems, one can conclude the Poincaré conjecture in the following way. Given M a simply connected, connected, compact Riemannian manifold, associate a Ricci flow with surgery. It has finite extinction time, and hence finite surgery times. Now one can use the topological decomposition to work backwards and build the manifold backwards, which says that the manifold M is the connected sum of finitely many spherical space forms, copies of $\mathbb{RP}^3\#\mathbb{RP}^3$, and $S^2\times S^1$. But since M is the simply connected, everything in the connected sum must be simply connected, and hence every piece of the connected sum must be simply connected, so M must be a sphere.