Quick Introduction to Riemannian geometry

David Glickenstein
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1 Introduction

We will try to get as quickly as possible to a point where we can do some geometric analysis on Riemannian spaces. One should look at Tao's lecture 0, though I will not follow it too closely.

2 Basics of tangent bundles and tensor bundles

Recall that for a smooth manifold M, the tangent bundle can be defined in essentially 3 different ways ((U_i, ϕ_i) are coordinates)

- $TM = \bigsqcup_i (U_i \times \mathbb{R}^n) / \sim$ where for $(x, v) \in U_i \times \mathbb{R}^n$, $(y, w) \in U_j \times \mathbb{R}^n$ we have $(x, v) \sim (y, w)$ if and only iff $y = \phi_j \phi_i^{-1}(x)$ and $w = d\left(\phi_j \phi_i^{-1}\right)_x (v)$.
- $T_pM = \{\text{paths } \gamma : (-\varepsilon, \varepsilon) \to M \text{ such that } \gamma(0) = p\} / \sim$ where $\alpha \sim \beta$ if $(\phi_i \circ \alpha)'(0) = (\phi_i \circ \beta)'(0)$ for every i such that $p \in U_i$. $TM = \bigsqcup_{p \in M} T_pM$.
- T_pM to be the set of derivations of germs at p, i.e., the set of linear functionals X on the germs at p such that $X(fg) = X(f)g(p) + f(p)X(g)$ for germs f, g at p. $TM = \bigsqcup_{p \in M} T_pM$.

On can define the cotangent bundle by essentially taking the dual of T_pM at each point, which we call T_p^*M, and taking the disjoint union of these to get the cotangent bundle T^*M. One could also use an analogue of the first definition, where the only difference is that instead of using the vector space $\mathbb{R}^n$, one uses its dual and the equivalence takes into account that the dual space pulls back rather than pushes forward. Both of these bundles are vector bundles. One can also take a tensor bundle of two vector bundles by replacing the fiber over a point by the tensor product of the fibers over the same point, e.g.,

$$TM \otimes T^*M = \bigsqcup_{p \in M} \left(T_pM \otimes T_p^*M\right).$$

Note that there are canonical isomorphisms of tensor products of vector spaces, such as $V \otimes V^*$ is isomorphic to endomorphisms of V. Note the difference between bilinear forms ($V^* \otimes V^*$), endomorphisms ($V \otimes V^*$), and bivectors ($V \otimes V$).

It is important to understand that these bundles are global objects, but will often be considered in coordinates. Given a coordinate $x = \left(x^i\right)$ and a point p in the coordinate patch, there is a basis $\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p$ for T_pM and dual basis $\left.dx^1\right|_p, \ldots, \left.dx^n\right|_p$ for T_p^*M. The generalization of the first definition above gives the idea of how one considers the trivializations of the bundle in a coordinate patch, and how the patches are linked together. Specifically, if x and y give different coordinates, for a point on the tensor bundle, one has

$$\begin{aligned}
&T_{ab\cdots c}^{ij\cdots k}(x)\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}\otimes\cdots\otimes\frac{\partial}{\partial x^k}\otimes dx^a\otimes dx^b\otimes\cdots\otimes dx^c\\
&=T_{ab\cdots c}^{ij\cdots k}(x(y))\left[\frac{\partial y^\zeta}{\partial x^i}\frac{\partial y^\eta}{\partial x^j}\cdots\frac{\partial y^\theta}{\partial x^k}\frac{\partial x^a}{\partial y^\alpha}\frac{\partial x^b}{\partial y^\beta}\cdots\frac{\partial x^c}{\partial y^\gamma}\right]\frac{\partial}{\partial y^\zeta}\otimes\frac{\partial}{\partial y^\eta}\otimes\cdots\otimes\frac{\partial}{\partial y^\theta}\otimes dy^\alpha\otimes dy^\beta\otimes\cdots\otimes dy^\gamma,
\end{aligned}$$

where technically everything should be at p (but as we shall see, one can consider this for all points in the neighborhood and this is considered as an equation of sections). Recall that a *section* of a bundle $\pi : E \to B$ is a function $f : B \to E$ such that $\pi \circ f$ is the identity on the base manifold B. A local section may only be defined on an open set in B. On the tangent space, sections are called *vector fields* and on the cotangent space, sections are called *forms* (or 1-forms). On a tensor bundle, sections are called *tensors*. Note that the set of $\frac{\partial}{\partial x^i}$ form a basis for the vector fields in the coordinate x, and dx^i form a basis for the local 1-forms in the coordinates. Sections in general are often written as $\Gamma(E)$ or as $C^\infty(E)$ (if we are considering smooth sections).

Now the equation above makes sense as an equation of tensors (sections of a tensor bundle). Often, a tensor will be denoted as simply

$$T_{ab\cdots c}^{ij\cdots k}.$$

Note that if we change coordinates, we have a different representation $T_{\alpha\beta\cdots\gamma}^{\zeta\eta\cdots\theta}$ of the same tensor. The two are related by

$$T_{\alpha\beta\cdots\gamma}^{\zeta\eta\cdots\theta}=T_{ab\cdots c}^{ij\cdots k}(x(y))\left[\frac{\partial y^\zeta}{\partial x^i}\frac{\partial y^\eta}{\partial x^j}\cdots\frac{\partial y^\theta}{\partial x^k}\frac{\partial x^a}{\partial y^\alpha}\frac{\partial x^b}{\partial y^\beta}\cdots\frac{\partial x^c}{\partial y^\gamma}\right].$$

One can also take subsets or quotients of a tensor bundle. In particular, we may consider the set of symmetric 2-tensors or anti-symmetric tensors (sections of this bundle are called differential forms). In particular, we have the Riemannian metric tensor.

Definition 1 *A Riemannian metric g is a two-tensor (i.e., a section of $T^*M \otimes T^*M$) which is*

- *symmetric, i.e., $g(X,Y) = g(Y,X)$ for all $X, Y \in T_pM$, and*

- *positive definite, i.e.,* $g(X,X) \geq 0$ *all* $X \in T_pM$ *and* $g(X,X) = 0$ *if and only if* $X = 0$.

Often, we will denote the metric as g_{ij}, which is shorthand for $g_{ij}dx^i dx^j$, where $dx^i dx^j = \frac{1}{2}\left(dx^i \otimes dx^j + dx^j \otimes dx^i\right)$. Note that if $g_{ij} = \delta_{ij}$ (the Kronecker delta) then

$$\delta_{ij}dx^i dx^j = \left(dx^1\right)^2 + \cdots \left(dx^n\right)^2.$$

One can invariantly define a trace of an endomorphism (trace of a matrix) which is independent of the coordinate change, since

$$\begin{aligned}\sum_{a=1}^{n} T_a^a &= \sum_a T_\beta^\alpha \frac{\partial x^a}{\partial y^\alpha}\frac{\partial y^\beta}{\partial x^a} \\ &= \sum_a T_\beta^\alpha \delta_\alpha^\beta \\ &= \sum_\alpha T_\alpha^\alpha.\end{aligned}$$

In fact for any complicated tensor, one can take the trace in one up index and one down index. This is called contraction. Usually, when there is a repeated index of one up and one down, we do not write the sum. This is called Einstein summation convention. The above sum would be written

$$T_a^a = T_\alpha^\alpha.$$

It is understood that this is an equation of functions.

We cannot contract two indices up or two indices down, since this is not independent of coordinate change (try it!) However, now that we have the Riemannian metric, we can use it to "lower an index" and then trace, so we get

$$T^{ab}g_{ba} = T_a^a.$$

In order to raise the index, we need the dual to the Riemannian metric, which is g^{ab}, defined such that $g^{ab}g_{bc} = \delta_c^a$ (so g^{ab} is the inverse matrix of g_{ab}). Then we can use g^{ab} to raise indices and contract if necessary. Occasionally, extended Einstein convention is used, where all repeated indices are summed with the understanding that the Riemannian metric is used to raise or lower indices when necessary, e.g.,

$$T_{aa} = T_{ab}g^{ab}.$$

Since often we will be changing the Riemannian metric, it becomes important to understand that the metric is there when extended Einstein is used.

3 Connections and covariant derivatives

3.1 What is a connection?

A covariant derivative is a particular way of differentiating vector fields. Why do we need a new way to differentiate vector fields? Here is the idea. Suppose we want to give a notion of parallel vectors. In $\mathbb{R}^n$, we know that if we take vector fields with constant coefficients, those vectors are parallel at different points. That is, the vectors $\frac{\partial}{\partial x^1}\big|_{(0,0)}+2\frac{\partial}{\partial x^2}\big|_{(0,0)}$ and $\frac{\partial}{\partial x^1}\big|_{(1,-1)}+2\frac{\partial}{\partial x^2}\big|_{(1,-1)}$ are parallel. In fact, we could say that the vector field $\frac{\partial}{\partial x^1}+2\frac{\partial}{\partial x^2}$ is parallel since vectors at any two points are parallel. One might say it is because the coefficients of the vector field are constant (not functions of x^1 and x^2). However, this notion is not invariant under a change of coordinates. Suppose we consider the new coordinates $\left(y^1,y^2\right)=\left(x^1,\left(x^2\right)^2\right)$ away from $x^2=0$ (where it is not a diffeomorphism). Then the vector field in the new coordinates is

$$\frac{\partial y^i}{\partial x^1}\frac{\partial}{\partial y^i}+2\frac{\partial y^j}{\partial x^2}\frac{\partial}{\partial x^j}=\frac{\partial}{\partial y^1}+4x^2\frac{\partial}{\partial y^2}=\frac{\partial}{\partial y^1}+4\sqrt{y^2}\frac{\partial}{\partial y^2}.$$

The coefficients are not constant, but the vector field should still be parallel (we have only changed coordinates, so it is the same vector field)! So we need a notion of parallel vector field that is independent of coordinate changes (or covariant).

Remember that we want to generalize the notion that a vector field has constant coefficients. Let $X=X^i\frac{\partial}{\partial x^i}$ be a vector field in a coordinate patch. Roughly speaking, we want to generalize the notion that $\frac{\partial X^i}{\partial x^j}=0$ for all i and j. The problem occurred because $\frac{\partial}{\partial x^1}\left(\frac{\partial}{\partial x^1}\right)$ is different in different coordinates. Thus we need to specify what this is. Certainly, since $\frac{\partial}{\partial x^i}$ is a basis, we must get a linear combination of these, so we take

$$\nabla_i\frac{\partial}{\partial x^j}=\Gamma_{ij}^k\frac{\partial}{\partial x^k}$$

for some functions Γ_{ij}^k. These symbols are called Christoffel symbols. To make sense on a vector field, we must have

$$\begin{aligned}\nabla_i\left(X\right)&=\nabla_i\left(X^j\frac{\partial}{\partial x^j}\right)\\&=\frac{\partial X^j}{\partial x^i}\frac{\partial}{\partial x^j}+X^j\Gamma_{ij}^k\frac{\partial}{\partial x^k}\\&=\left(\frac{\partial X^k}{\partial x^i}+X^j\Gamma_{ij}^k\right)\frac{\partial}{\partial x^k}.\end{aligned}$$

Notice the Leibniz rule (product rule). One can now define ∇ for any vector $Y=Y^i\frac{\partial}{\partial x^i}$ by

$$\nabla_YX=\nabla_{Y^i\frac{\partial}{\partial x^i}}X=Y^i\left(\nabla_iX\right).$$

This action is called the covariant derivative.

One now defines Γ_{ij}^k in such a way that the covariant derivative transforms appropriately under change of coordinates. This gives a global object called a connection. The connection can be defined axiomatically as follows.

Definition 2 *A connection on a vector bundle $E \to B$ is a map*

$$\nabla : \Gamma(TB) \otimes \Gamma(E) \to \Gamma(E)$$

$$(X, \phi) \to \nabla_X \phi$$

satisfying:

- *Tensoriality (i.e., $C^\infty(B)$-linear) in the first component, i.e., $\nabla_{fX+Y}\phi = f\nabla_X\phi + \nabla_Y\phi$ for any function f and vector fields X, Y*
- *Derivation in the second component, i.e., $\nabla_X(f\phi) = X(f)\phi + f\nabla_X\phi$.*
- *$\mathbb{R}$-linear in the second component, i.e., $\nabla_X(a\phi + \psi) = a\nabla_X(\phi) + \nabla_X(\psi)$ for $a \in \mathbb{R}$.*

We will consider connections primarily on the tangent bundle and tensor bundles. Note that a connection ∇ on TM induces connections on all tensor bundles (also denoted ∇) in the following way:

- For a function f and vector field X, we define $\nabla_X f = Xf$
- For vector fields X, Y and dual form ω, we use the product rule to derive

  $$\nabla_X(\omega(Y)) = X(\omega(Y)) = (\nabla_X\omega)(Y) + \omega(\nabla_X Y)$$

  and thus

  $$(\nabla_X\omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y).$$

  In particular, the Christoffel symbols for the connection on T^*M are the negative of the Christoffel symbols of TM, i.e.,

  $$\nabla_{\frac{\partial}{\partial x^i}} dx^j = -\Gamma^j_{ik} dx^k$$

  where Γ^k_{ij} are the Christoffel symbols for the connection ∇ on TM.
- For a tensor product, one defines the connection using the product rule, e.g.,

  $$\nabla_X(Y \otimes \omega) = (\nabla_X Y) \otimes \omega + Y \otimes \nabla_X \omega$$

  for vector fields X, Y and 1-form ω.

Remark 3 *The Christoffel symbols are not tensors. Note that if we change coordinates from x to $\tilde{x}$, we have*

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \nabla_{\left(\frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial}{\partial \tilde{x}^k}\right)} \left(\frac{\partial \tilde{x}^\ell}{\partial x^j}\frac{\partial}{\partial \tilde{x}^\ell}\right) = \frac{\partial \tilde{x}^\ell}{\partial x^i \partial x^j}\frac{\partial}{\partial \tilde{x}^\ell} + \frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial \tilde{x}^\ell}{\partial x^j}\nabla_{\frac{\partial}{\partial \tilde{x}^k}}\frac{\partial}{\partial \tilde{x}^\ell}$$

which means that

$$\Gamma^k_{ij} = \tilde{\Gamma}^m_{p\ell}\frac{\partial \tilde{x}^p}{\partial x^i}\frac{\partial \tilde{x}^\ell}{\partial x^j}\frac{\partial x^k}{\partial \tilde{x}^m} + \frac{\partial \tilde{x}^\ell}{\partial x^i \partial x^j}\frac{\partial x^k}{\partial \tilde{x}^\ell}.$$

One final comment. Recall that we motivated the connection by considering parallel vector fields. The connection gives us a way of taking a vector at a point and translating it along a curve so that the induced vector field along the curve is parallel (i.e., $\nabla_{\dot{\gamma}} X = 0$ along γ). This is called *parallel translation.*

Parallel vector fields allow one to rewrite derivatives in coordinates; that is, if $X = X^i \frac{\partial}{\partial x^i}$ is parallel, then

$$\frac{\partial X^i}{\partial x^j} = -X^k \Gamma_{jk}^i.$$

3.2 Torsion, compatibility with the metric, and Levi-Civita connection

There is a unique metric associated with the Riemannian metric, called the Riemannian connection or Levi-Civita connection. It satisfies two properties:

- Torsion-free (also called symmetric)
- Compatible with the metric.

Compatibility with the metric is the easy one to understand. We want the connection to behave well with respect to differentiating orthogonal vector fields. Being compatible with the metric is the same as

$$\nabla_X \left(g \left(Y, Z \right) \right) = g \left(\nabla_X Y, Z \right) + g \left(Y, \nabla_X Z \right).$$

Note that normally there would be an extra term, $\left(\nabla_X g \right) \left(Y, Z \right)$, so compatibility with the metric means that this term is zero, i.e., $\nabla g = 0$, where g is considered as a 2-tensor.

Torsion free means that the torsion tensor τ, given by

$$\tau \left(X, Y \right) = \nabla_X Y - \nabla_Y X - [X, Y]$$

vanishes. (One can check that this is a tensor by verifying that $\tau \left(fX, Y \right) = \tau \left(X, fY \right) = f \tau \left(X, Y \right)$ for any function f). It is easy to see that in coordinates, the torsion tensor is given by

$$\tau_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k,$$

which indicates why torsion-free is also called symmetric.

Tao gives a short motivation for the concept of torsion-free. Consider an infinitesimal parallelogram in the plane consisting of a point x, the flow of x along a vector field V to a point we will call $x + tV$, the flow of X along a vector field W to a point we will call $x + tW$, and then a fourth point which we will reach in two ways: (1) go to $x + tV$ and then flow along the parallel translation of W for a distance t and (2) go to $x + tW$ and then flow along the parallel translation of V for a distance t. Note that using method (1), we get that the point is

$$\begin{aligned}
&\left(x+tV+sW\right)|_{s=0}+t\left.\frac{\partial}{\partial s}\right|_{s=0}\left(x+tV+sW\right)+O\left(t^3\right)\\
&=x+tV+tW+t^2\left.\frac{\partial}{\partial s}\right|_{s=0}V+O\left(t^3\right)=x+tV+tW-t^2V^iW^j\Gamma_{ji}^k\frac{\partial}{\partial x^k}+O\left(t^3\right).
\end{aligned}$$

Note that using method (2), we get instead

$$x+tV+tW-t^2W^iV^j\Gamma_{ji}^k\frac{\partial}{\partial x^k}+O\left(t^3\right),$$

Thus this vector is $x+t\left(V+W\right)$ up to $O\left(t^3\right)$ only if $\Gamma_{ji}^k=\Gamma_{ij}^k$. Doing this around every infinitesimal parallelogram gives the equivalence of these two viewpoints.

Here is another:

Proposition 4 *A connection is torsion-free if and only if for any point $p\in M$, there are coordinates x around p such that $\Gamma_{ij}^k\left(p\right)=0$.*

Proof. Suppose one can always find coordinates such that $\Gamma_{ij}^k\left(p\right)=0$. Then clearly at that point, $\tau_{ij}^k=0$. However, since the torsion is a tensor, we can calculate it in any coordinate, so at each point, we have that the torsion vanishes. Now suppose the torsion tensor vanishes and let x be a coordinate around p. Consider the new coordinates

$$\tilde{x}^i\left(q\right)=x^i\left(q\right)-x^i\left(p\right)+\Gamma_{jk}^i\left(p\right)\left(x^j\left(q\right)-x^j\left(p\right)\right)\left(x^k\left(q\right)-x^k\left(p\right)\right).$$

Then notice that

$$\frac{\partial\tilde{x}^i}{\partial x^j}=\delta_j^i+\Gamma_{k\ell}^i\left(p\right)\delta_j^k\left(x^\ell-x^\ell\left(p\right)\right)+\Gamma_{k\ell}^i\left(p\right)\left(x^k-x^k\left(p\right)\right)\delta_j^\ell$$

and so

$$\frac{\partial\tilde{x}^i}{\partial x^j}\left(p\right)=\delta_j^i.$$

Thus $\tilde{x}$ is a coordinate patch in some neighborhood of p. Moreover, we have that

$$\frac{\partial^2\tilde{x}^i}{\partial x^j\partial x^k}=\Gamma_{jk}^i\left(p\right).$$

One can now verify that at p,

$$\begin{aligned}
\Gamma_{ij}^k\left(p\right)&=\tilde{\Gamma}_{p\ell}^m\frac{\partial\tilde{x}^p}{\partial x^i}\frac{\partial\tilde{x}^\ell}{\partial x^j}\frac{\partial x^k}{\partial\tilde{x}^m}+\frac{\partial\tilde{x}^\ell}{\partial x^i\partial x^j}\frac{\partial x^k}{\partial\tilde{x}^\ell}\\
&=\tilde{\Gamma}_{ij}^k\left(p\right)+\Gamma_{ij}^k\left(p\right).
\end{aligned}$$

■

The Riemannian connection is the unique connection which is both torsion-free and compatible with the metric. One can use these two properties to derive a formula for it. In coordinates, one finds that the Riemannian connection has the following Christoffel symbols

$$\Gamma_{ij}^{k}=\frac{1}{2}g^{k\ell}\left(\frac{\partial}{\partial x^{i}}g_{j\ell}+\frac{\partial}{\partial x^{j}}g_{i\ell}-\frac{\partial}{\partial x^{\ell}}g_{ij}\right).$$

One can easily verify that this connection has the properties expressed. Note that the $g_{j\ell}$ in the formula, etc. are not the tensors, but the functions. This is not a tensor equation since Γ_{ij}^{k} is not a tensor. Also note that it is very important that this is an expression in coordinates (i.e., that $\left[\frac{\partial}{\partial x^{i}},\frac{\partial}{\partial x^{j}}\right]=0$).

3.3 Higher derivatives of functions and tensors

One of the important reasons for having a connection is it allows us to take higher derivatives. Note that one can take the derivative of a function without a connection, and it is defined as

$$\begin{aligned} df &= \nabla f \\ df\left(X\right) &= \nabla_{X} f = X\left(f\right) \\ df &= \frac{\partial f}{\partial x^{i}}dx^{i}. \end{aligned}$$

One can also raise the index to get the gradient, which is

$$grad\left(f\right)=\nabla^{i}f\frac{\partial}{\partial x^{i}}=\frac{\partial f}{\partial x^{j}}g^{ij}\frac{\partial}{\partial x^{i}}.$$

However, to take the next derivative, one needs a connection. The second derivative, or *Hessian*, of a function is

$$\begin{aligned} Hess\left(f\right) &= \nabla^{2}f=\nabla df \\ \nabla^{2}f &= \left(\nabla_{i}df\right)\otimes dx^{i} \\ &= \left(\nabla_{i}\left(\frac{\partial f}{\partial x^{j}}dx^{j}\right)\right)\otimes dx^{i} \\ &= \left[\frac{\partial^{2}f}{\partial x^{i}\partial x^{j}}dx^{j}-\frac{\partial f}{\partial x^{j}}\Gamma_{ik}^{j}dx^{k}\right]\otimes dx^{i} \\ &= \left(\frac{\partial^{2}f}{\partial x^{i}\partial x^{j}}-\frac{\partial f}{\partial x^{k}}\Gamma_{ij}^{k}\right)dx^{j}\otimes dx^{i}. \end{aligned}$$

Often one will write the Hessian as

$$\nabla_{ij}^{2}f=\nabla_{i}\nabla_{j}f=\frac{\partial^{2}f}{\partial x^{i}\partial x^{j}}-\Gamma_{ij}^{k}\frac{\partial f}{\partial x^{k}}.$$

Note that if the connection is symmetric, then the Hessian of a function is symmetric in the usual sense. The trace of the Hessian, $\triangle f=g^{ij}\nabla_{ij}^{2}f$, is called the Laplacian, and we will use it quite a bit.

We also may use the connection to compute acceleration of a curve. The velocity of a curve is $\dot{\gamma}$, which does not need a connection, but to compute the *acceleration*, $\nabla_{\dot{\gamma}}\dot{\gamma}$, we need the connection (one also sometimes sees the equivalent notation $D\dot{\gamma}/dt$). A curve with zero acceleration is called a *geodesic*.

Finally, given any tensor T, one can use the connection to form a new tensor ∇T, which has an extra down index.

4 Curvature

One can define the curvature of any connection on a bundle $E \to B$ in the following way

$$R : \Gamma(TM) \otimes \Gamma(TM) \otimes \Gamma(E) \to \Gamma(E)$$
$$R(X,Y)\,\phi = \nabla_X \nabla_Y \phi - \nabla_Y \nabla_X \phi - \nabla_{[X,Y]}\phi.$$

We will consider the curvature of the Riemannian connection on the tangent bundle. One can easily see that in coordinates, the curvature is a tensor denoted as

$$\nabla_i \nabla_j \frac{\partial}{\partial x^k} - \nabla_j \nabla_i \frac{\partial}{\partial x^k} = R_{ijk}^{\ell} \frac{\partial}{\partial x^\ell}$$

which gives us that

$$\begin{aligned}
\nabla_i \left(\Gamma_{jk}^{\ell} \frac{\partial}{\partial x^\ell} \right) - \nabla_j \left(\Gamma_{ik}^{\ell} \frac{\partial}{\partial x^\ell} \right) &= \left(\frac{\partial}{\partial x^i} \Gamma_{jk}^{\ell} \right) \frac{\partial}{\partial x^\ell} + \Gamma_{jk}^{\ell} \Gamma_{i\ell}^{m} \frac{\partial}{\partial x^m} - \left(\frac{\partial}{\partial x^j} \Gamma_{ik}^{\ell} \right) \frac{\partial}{\partial x^\ell} - \Gamma_{ik}^{\ell} \Gamma_{j\ell}^{m} \frac{\partial}{\partial x^m} \\
&= \left(\frac{\partial}{\partial x^i} \Gamma_{jk}^{\ell} - \frac{\partial}{\partial x^j} \Gamma_{ik}^{\ell} + \Gamma_{jk}^{m} \Gamma_{im}^{\ell} - \Gamma_{ik}^{m} \Gamma_{jm}^{\ell} \right) \frac{\partial}{\partial x^\ell}
\end{aligned}$$

So the curvature tensor is

$$R_{ijk}^{\ell} = \frac{\partial}{\partial x^i} \Gamma_{jk}^{\ell} - \frac{\partial}{\partial x^j} \Gamma_{ik}^{\ell} + \Gamma_{jk}^{m} \Gamma_{im}^{\ell} - \Gamma_{ik}^{m} \Gamma_{jm}^{\ell}.$$

Often we will lower the index, and consider instead the curvature tensor

$$R_{ijk\ell} = R_{ijk}^{m} g_{m\ell}.$$

The Riemannian curvature tensor has the following symmetries:

- $R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij}$ (These imply that R can be viewed as a self-adjoint (symmetric) operator mapping 2-forms to 2-forms if one raises the first two or last two indices).
- (Algebraic Bianchi) $R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0$.
- (Differential Bianchi) $\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m}$.

Remark 5 *The tensor $R_{ijk\ell}$ can also be written as a tensor $R(X,Y,Z,W)$, which is a function when vector fields X,Y,Z,W are plugged in. We will sometimes refer to this tensor as* Rm. *The tensor R_{ijk}^{ℓ} is usually denoted by $R(X,Y)Z$, which is a vector field when vector fields X,Y,Z are plugged in.*

Remark 6 *Sometimes, the up index is lowered into the 3rd spot instead of the 4th, This will change the definitions of Ricci and sectional curvature below, but the sectional curvature of the sphere should always be positive and the Ricci curvature of the sphere should be positive definite.*

Remark 7 *Note that Γ_{ij}^k involved first derivatives of the metric, so Riemannian curvature tensor involves first and second derivatives of the metric.*

From these one can derive all the curvatures we will need:

Definition 8 *The Ricci curvature tensor R_{ij} is defined as*

$$R_{ij} = R_{\ell ij}^{\ell} = R_{\ell ijm} g^{\ell m}.$$

Note that $R_{ij} = R_{ji}$ by the symmetries of the curvature tensor. Ricci will sometimes be denoted $\operatorname{Rc}(g)$, *or* $\operatorname{Rc}(X,Y)$.

Definition 9 *The scalar curvature R is the function*

$$R = g^{ij} R_{ij}$$

Definition 10 *The sectional curvature of a plane spanned by vectors X and Y is given by*

$$K(X,Y) = \frac{R(X,Y,Y,X)}{g(X,X)\,g(Y,Y) - g(X,Y)^2}.$$

Here are some facts about the curvatures:

Proposition 11
1. *The sectional curvatures determine the entire curvature tensor, i.e., if one can calculate all sectional curvatures, then one can calculate the entire tensor.*
2. *The sectional curvature $K(X,Y)$ is the Gaussian curvature of the surface generated by geodesics in the plane spanned by X, Y.*
3. *The Ricci curvature can be written as an average of sectional curvature.*
4. *The scalar curvature can be written as an average of Ricci curvatures.*
5. *The scalar curvature essentially gives the difference between the volumes of small metric balls and the volumes of Euclidean balls of the same radius.*
6. *In 2 dimensions, each curvature determines the others.*
7. *In 3 dimensions, scalar curvature does not determine Ricci, but Ricci does determine the curvature tensor.*
8. *In dimensions larger than 3, Ricci does not determine the curvature tensor; there is an additional piece called the Weyl tensor.*

With this in mind, we can talk about several different kinds of nonnegative curvature.

Definition 12 *Let x be a point on a Riemannian manifold (M, g). Then x has*

1. *nonnegative scalar curvature if $R(x) \geq 0$;*
2. *nonnegative Ricci curvature at x if $\mathrm{Rc}(X, X) = R_{ij} X^i X^j \geq 0$ for every vector $X \in T_x M$;*
3. *nonnegative sectional curvature if $R(X, Y, Y, X) = g(R(X, Y) Y, X) \geq 0$ for all vectors $X, Y \in T_x M$;*
4. *nonnegative Riemann curvature (or nonnegative curvature operator) if $\mathrm{Rm} \geq 0$ as a quadratic form on $\Omega^2(M)$, i.e., if $R_{ijk\ell}\omega^{ij}\omega^{k\ell} \geq 0$ for all 2-forms $\omega = \omega_{ij} dx^i \wedge dx^j$ (where the raised indices are done using the metric g).*

It is not too hard to see that 4 implies 3 implies 2 implies 1. Also, in 3 dimensions, 3 and 4 are equivalent. In dimension 4 and higher, these are all distinct.