Introduction to flows on Riemannian metrics

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1 Introduction

This lecture roughly follows Tao's Lecture 1. We will talk in general about flows or Riemannian metrics and Ricci flow.

We will consider a flow of Riemannian metrics to be a one-parameter family of Riemannian metrics, usually denoted $g(t)$ or $g_{ij}(t)$ or $g_{ij}(x,t)$ on a fixed Riemannian manifold M. There are more ingenious ways to define such a flow using spacetimes (called generalized Ricci flows). However, at present I do not think that they give a significant savings over the more classical idea, since one still needs to consider singular spacetimes. For more on generalized Ricci flows, consult the book by Morgan-Tian.

The family $g(t)$ is a one-parameter family of sections of a vector bundle, and one can take its derivative as

$$\frac{\partial}{\partial t} g(t) = \lim_{dt \to 0} \frac{g(t+dt) - g(t)}{dt}$$

since $g(t)$ and $g(t+dt)$ are both sections of the same vector bundle, so the difference makes sense. In fact, we can differentiate any tensor in this way. Similarly, we can try to solve differential equations of the form

$$\frac{\partial}{\partial t} g_{ij} = \dot{g}_{ij}$$

for some prescribed $\dot{g}_{ij}$. The evolution of the metric induces an evolution of the metric on the cotangent bundle, using

$$\begin{aligned} \frac{\partial}{\partial t}\left(g^{ij} g_{jk}\right) &= \frac{\partial}{\partial t} \delta^i_j \\ \frac{\partial}{\partial t} g^{ij} &= -g^{ik} \dot{g}_{k\ell} g^{\ell j}. \end{aligned}$$

The Riemannian connection is also changing if the metric is changing. Thus for a fixed vector field X, we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_i X &= \frac{\partial}{\partial t}\left[\frac{\partial X^j}{\partial x^i}\frac{\partial}{\partial x^j} + \Gamma_{ij}^k X^j \frac{\partial}{\partial x^k}\right] \\ &= X^j \dot{\Gamma}_{ij}^k \frac{\partial}{\partial x^k}.\end{aligned}$$

We can use the fact that the connection is torsion-free and compatible with the metric to derive the formula for $\dot{\Gamma}_{ij}^k$:

$$\begin{aligned}0 &= \frac{\partial}{\partial t}\left(\nabla_i g_{jk}\right) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x^i} g_{jk} - \Gamma_{ij}^{\ell} g_{\ell k} - \Gamma_{ik}^{\ell} g_{j\ell}\right) \\ &= \nabla_i \dot{g}_{jk} - \dot{\Gamma}_{ij}^{\ell} g_{\ell k} - \dot{\Gamma}_{ik}^{\ell} g_{j\ell},\end{aligned}$$

and

$$0 = \dot{\Gamma}_{ij}^k - \dot{\Gamma}_{ji}^k$$

so we can solve for $\dot{\Gamma}_{ij}^k$ as

$$\begin{aligned}\nabla_i \dot{g}_{jk} &= \dot{\Gamma}_{ij}^{\ell} g_{\ell k} + \dot{\Gamma}_{ik}^{\ell} g_{j\ell} \\ \nabla_j \dot{g}_{ki} &= \dot{\Gamma}_{jk}^{\ell} g_{\ell k} + \dot{\Gamma}_{ji}^{\ell} g_{k\ell} \\ \nabla_k \dot{g}_{ij} &= \dot{\Gamma}_{ki}^{\ell} g_{\ell j} + \dot{\Gamma}_{kj}^{\ell} g_{i\ell}\end{aligned}$$

to get

$$\dot{\Gamma}_{ij}^k = \frac{1}{2} g^{k\ell}\left(\nabla_i \dot{g}_{j\ell} + \nabla_j \dot{g}_{i\ell} - \nabla_\ell \dot{g}_{ij}\right). \tag{1}$$

Remark 1 *This mimics the proof of the formula for the Riemannian connection given that it is torsion-free and compatible with the metric. There are other ways to derive this formula, for instance by computing in normal coordinates and using the fact that although Γ_{ij}^k is not a tensor, $\frac{\partial}{\partial t}\Gamma_{ij}^k$ comes from the difference of two connections and is thus a tensor. We will use this method below.*

We may now look at the induced formula for evolution of the Riemannian curvature tensor. Recall that, in coordinates,

$$R_{ijk}^{\ell}\frac{\partial}{\partial x^\ell} = \nabla_i\left(\Gamma_{jk}^{\ell}\frac{\partial}{\partial x^\ell}\right) - \nabla_j\left(\Gamma_{ik}^{\ell}\frac{\partial}{\partial x^\ell}\right).$$

Since we are interested in the derivative of a tensor, $\frac{\partial}{\partial t}R_{ijk}^{\ell} = \dot{R}_{ijk}^{\ell}$, we can compute this in any coordinate system we want. Recall that there is a coordinate system around p such that all Christoffel symbols vanish at p. Doing this reduces the equation to

$$
\begin{aligned}
\dot{R}^{\ell}_{ijk}\frac{\partial}{\partial x^{\ell}} &= \frac{\partial}{\partial t}\left[\nabla_i\left(\Gamma^{\ell}_{jk}\frac{\partial}{\partial x^{\ell}}\right) - \nabla_j\left(\Gamma^{\ell}_{ik}\frac{\partial}{\partial x^{\ell}}\right)\right] \\
&= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x^i}\Gamma^{\ell}_{jk}\frac{\partial}{\partial x^{\ell}} - \frac{\partial}{\partial x^j}\Gamma^{\ell}_{ik}\frac{\partial}{\partial x^{\ell}}\right) \\
&= \frac{\partial}{\partial x^i}\dot{\Gamma}^{\ell}_{jk}\frac{\partial}{\partial x^{\ell}} - \frac{\partial}{\partial x^j}\dot{\Gamma}^{\ell}_{ik}\frac{\partial}{\partial x^{\ell}} \\
&= \nabla_i\dot{\Gamma}^{\ell}_{jk}\frac{\partial}{\partial x^{\ell}} - \nabla_j\dot{\Gamma}^{\ell}_{ik}\frac{\partial}{\partial x^{\ell}}.
\end{aligned}
$$

This last piece is tensorial (recall that $\dot{\Gamma}^k_{ij}$ is a tensor), and thus only depends on the point, not the coordinate patch, so we must have that

$$\dot{R}^{\ell}_{ijk} = \nabla_i\dot{\Gamma}^{\ell}_{jk} - \nabla_j\dot{\Gamma}^{\ell}_{ik}.$$

We can now use the (1) to get

$$
\begin{aligned}
\dot{R}^{\ell}_{ijk} &= \nabla_i\left(\frac{1}{2}g^{\ell m}\left(\nabla_j\dot{g}_{km} + \nabla_k\dot{g}_{jm} - \nabla_m\dot{g}_{jk}\right)\right) - \nabla_j\left(\frac{1}{2}g^{\ell m}\left(\nabla_i\dot{g}_{km} + \nabla_k\dot{g}_{im} - \nabla_m\dot{g}_{ik}\right)\right) \\
&= \frac{1}{2}g^{\ell m}(\nabla_i\nabla_j\dot{g}_{km} + \nabla_i\nabla_k\dot{g}_{jm} - \nabla_i\nabla_m\dot{g}_{jk} - \nabla_j\nabla_i\dot{g}_{km} - \nabla_j\nabla_k\dot{g}_{im} + \nabla_j\nabla_m\dot{g}_{ik}) \\
&= \frac{1}{2}g^{\ell m}(\nabla_i\nabla_j\dot{g}_{km} - \nabla_j\nabla_i\dot{g}_{km} + \nabla_i\nabla_k\dot{g}_{jm} - \nabla_i\nabla_m\dot{g}_{jk} - \nabla_j\nabla_k\dot{g}_{im} + \nabla_j\nabla_m\dot{g}_{ik}) \\
&= \frac{1}{2}g^{\ell m}(-R^p_{ijk}\dot{g}_{mp} - R^p_{ijm}\dot{g}_{kp} + \nabla_i\nabla_k\dot{g}_{jm} - \nabla_i\nabla_m\dot{g}_{jk} - \nabla_j\nabla_k\dot{g}_{im} + \nabla_j\nabla_m\dot{g}_{ik}).
\end{aligned}
$$

We can take the trace $\dot{R}_{jk} = \dot{R}^i_{ijk}$ to get

$$
\begin{aligned}
\dot{R}_{jk} &= \frac{1}{2}g^{im}(-R^p_{ijk}\dot{g}_{mp} - R^p_{ijm}\dot{g}_{kp} + \nabla_i\nabla_k\dot{g}_{jm} - \nabla_i\nabla_m\dot{g}_{jk} - \nabla_j\nabla_k\dot{g}_{im} + \nabla_j\nabla_m\dot{g}_{ik}) \\
&= -\frac{1}{2}g^{im}R^p_{ijk}\dot{g}_{mp} + \frac{1}{2}g^{pq}R_{jp}\dot{g}_{kp} - \frac{1}{2}g^{im}\nabla_i\nabla_m\dot{g}_{jk} - \frac{1}{2}g^{im}\nabla_j\nabla_k\dot{g}_{im} \\
&\quad + \frac{1}{2}g^{im}(\nabla_k\nabla_i\dot{g}_{jm} - R^p_{ikj}\dot{g}_{pm} - R^p_{ikm}\dot{g}_{jp} + \nabla_j\nabla_m\dot{g}_{ik}) \\
&= -g^{im}R^p_{ijk}\dot{g}_{mp} + \frac{1}{2}g^{pq}R_{jp}\dot{g}_{kp} + \frac{1}{2}g^{pq}R_{kq}\dot{g}_{jp} - \frac{1}{2}g^{im}\nabla_i\nabla_m\dot{g}_{jk} \\
&\quad - \frac{1}{2}g^{im}\nabla_j\nabla_k\dot{g}_{im} + \frac{1}{2}g^{im}(\nabla_k\nabla_i\dot{g}_{jm} + \nabla_j\nabla_m\dot{g}_{ik}) \\
&= -\frac{1}{2}\triangle_L\dot{g}_{jk} - \frac{1}{2}g^{im}\nabla_j\nabla_k\dot{g}_{im} + \frac{1}{2}g^{im}(\nabla_k\nabla_i\dot{g}_{jm} + \nabla_j\nabla_m\dot{g}_{ik})
\end{aligned}
$$

where

$$\triangle_L\dot{g}_{jk} = g^{im}\nabla_i\nabla_m\dot{g}_{jk} + 2g^{im}R^p_{ijk}\dot{g}_{mp} - g^{pq}R_{jp}\dot{g}_{kp} - g^{pq}R_{kq}\dot{g}_{jp}$$

is the Lichnerowitz Laplacian (notice only the first term has two derivatives of $\dot{g}_{jk}$). (Note: I think that T. Tao has an error in this formula with the sign of the last term of $\dot{R}_{jk}$.

Finally, we may take a trace to get

$$\begin{aligned}\dot{R} &= g^{jk}\dot{R}_{jk} - g^{pj}\dot{g}_{pq}g^{qk}R_{jk} \\ &= -g^{pq}g^{jk}R_{jp}\dot{g}_{kp} - g^{im}\nabla_i\nabla_m g^{jk}\dot{g}_{jk} \\ &\quad + g^{im}g^{jk}\nabla_k\nabla_i\dot{g}_{jm} \\ &= -\langle \mathrm{Rc}, \dot{g}\rangle - \triangle \operatorname{tr}_g(\dot{g}) + \operatorname{div}\operatorname{div}\dot{g}\end{aligned}$$

where

$$(\operatorname{div} h)_j = g^{k\ell}\nabla_k h_{\ell j}$$

2 Ricci flow

Note that in the evolution of Ricci curvature, if one considers

$$\dot{g} = -2\,\mathrm{Rc},$$

one gets

$$\begin{aligned}&-\frac{1}{2}g^{im}\nabla_j\nabla_k\dot{g}_{im} + \frac{1}{2}g^{im}(\nabla_k\nabla_i\dot{g}_{jm} + \nabla_j\nabla_m\dot{g}_{ik}) \\ &= g^{im}\nabla_j\nabla_k R_{im} - g^{im}\nabla_k\nabla_i R_{jm} - g^{im}\nabla_j\nabla_m R_{ik} \\ &= \nabla_j\nabla_k R - g^{im}\nabla_j\nabla_m R_{ik} - g^{im}\nabla_k\nabla_i R_{jm}\end{aligned}$$

Note that the differential Bianchi identity implies

$$\begin{aligned}0 &= g^{jm}g^{k\ell}\left(\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m}\right) \\ &= \nabla_i R - g^{jm}\nabla_j R_{im} - g^{k\ell}\nabla_k R_{ij\ell m}\end{aligned}$$

so

$$\nabla_i R = 2g^{jm}\nabla_j R_{im}$$

so

$$-\frac{1}{2}g^{im}\nabla_j\nabla_k\dot{g}_{im} + \frac{1}{2}g^{im}(\nabla_k\nabla_i\dot{g}_{jm} + \nabla_j\nabla_m\dot{g}_{ik}) = \nabla_j\nabla_k R - \frac{1}{2}\nabla_j\nabla_k R - \frac{1}{2}\nabla_k\nabla_j R = 0.$$

Thus under Ricci flow,

$$\frac{\partial}{\partial t}\mathrm{Rc} = \triangle_L \mathrm{Rc}\,.$$

Furthermore, We see that

$$\begin{aligned}\frac{\partial R}{\partial t} &= 2\langle \mathrm{Rc}, \mathrm{Rc}\rangle + 2\triangle R - 2g^{i\ell}g^{jk}\nabla_i\nabla_j R_{k\ell} \\ &= -2\langle \mathrm{Rc}, \mathrm{Rc}\rangle + 2\triangle R - g^{i\ell}\nabla_i\nabla_\ell R \\ &= \triangle R + 2\left|\mathrm{Rc}\right|^2\end{aligned}$$

The important notion to get right now is that this looks very much like a heat equation with a reaction term. We will see how to make use of this in the near future.

3 Existence/Uniqueness

Note that the Ricci flow equation,

$$\frac{\partial}{\partial t} g = -2 \operatorname{Rc}$$

is a second order partial differential equation, since the Ricci curvature comes from second derivatives of the metric. To truly look at existence/uniqueness, one must write this as an equation in coordinates. We will look at the linearization of this operator in order to find the principle symbol (which is basically the coefficients of the linearization of the highest derivatives). Analysis of the principle symbol will often allow us to determine that a solution exists for a short time. Here is the meta-theorem for existence of parabolic PDE:

Meta-Theorem (imprecise): *A semi-linear PDE of the form*

$$\frac{\partial u}{\partial t} - a^{ij}(x,t) \frac{\partial^2 u}{\partial x^i \partial x^j} + F(x,t,u,\partial u) = 0.$$

on a compact manifold has a solution with initial condition $u(x,0) = f(x)$ if there exists $\delta > 0$ such that $a^{ij}\xi_i\xi_j \geq \delta |\zeta|^2$ (this condition is called strict parabolicity) for t close to 0. Similarly, if we allow a^{ij} to depend on u (making the equation quasilinear, the same is true if we look at the linearization (which is then semilinear).

Remark 2 *a^{ij} is called the principal symbol of the parabolic differential operator. If one takes out the $\frac{\partial}{\partial t}$, the differential operator is said to be elliptic if it satisfies the inequality.*

Remark 3 *We can replace $\frac{\partial}{\partial x^i}$ with ∇_i since the difference has fewer derivatives.*

Remark 4 *One should be able to prove a coordinate independent version, but this is not usually done. All theory is based on theory of differential equations on domains in the plane.*

Remark 5 *For an arbitrary, nonlinear second order PDE of the form*

$$G\left(x,t,u,\partial u,\partial^2 u\right) = 0,$$

one can consider the linearization of G with respect to u. This will look roughly like

$$\left[\partial_{H_{ij}} G \frac{\partial^2}{\partial x^i \partial x^j} + \partial_{V_i} G \frac{\partial}{\partial x^i} + \partial_u G\right] v$$

where $G = G(x,t,u,V,H)$ and the operator is evaluated at some u (which is where it has been linearized. Notice this now gives a semilinear PDE. The principle symbol is $\partial_{H_{ij}} G\xi_i\xi_j$.

Example 6 *Note that the equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\left(\partial x^1\right)^2} u + \frac{\partial^2}{\left(\partial x^2\right)^2} u = \delta^{ij} \frac{\partial^2}{\partial x^i \partial x^j}$$

is already linear. For an equation like

$$\frac{\partial u}{\partial t} = u^2 \frac{\partial^2 u}{\partial x^2},$$

the linearization is

$$\frac{\partial v}{\partial t} = u^2 \frac{\partial^2 v}{\partial x^2} + 2u \frac{\partial^2 u}{\partial x^2} v,$$

thus the principal symbol is u^2 which is positive if $u > 0$.

Now, the Ricci operator is an operator on sections, not just functions, so how do we make sense of the kind of result given above. We can make a similar definition in terms of the linearization, but now the principle symbol is a map from sections of the symmetric 2-tensor bundle to itself. What we need is that for any $\xi \neq 0$, the principle symbol is a linear isomorphism.

Recall that the linearization of R_{jk} is

$$\dot{R}_{jk} = -\frac{1}{2} g^{im} \nabla_i \nabla_m \dot{g}_{jk} - \frac{1}{2} g^{im} \nabla_j \nabla_k \dot{g}_{im} + \frac{1}{2} g^{im} (\nabla_k \nabla_i \dot{g}_{jm} + \nabla_j \nabla_m \dot{g}_{ik})$$

so the principle symbol of $-2R_{jk}$ is

$$\hat{\sigma}\left[D\,\mathrm{Rc}\right](\xi)(h) = g^{im} \xi_i \xi_m h_{jk} + g^{im} \xi_j \xi_k h_{im} - g^{im} (\xi_k \xi_i h_{jm} + \xi_j \xi_k h_{ik}).$$

In order to see if this is an isomorphism, we can rotate ξ so that $\xi_1 > 0$ and $\xi_2 = \cdots = \xi_n = 0$ and by scaling we can assume $\xi_1 = 1$. We can also assume that at a point $g_{ij} = \delta_{ij}$. Then we see that

$$\hat{\sigma}\left[D\,\mathrm{Rc}\right](\xi)(h)_{jk} = h_{jk} + \delta_j^1 \delta_k^1 (h_{11} + \cdots + h_{nn}) - (\delta_k^1 h_{j1} + \delta_j^1 h_{1k}).$$

And so the matrix for the symbol gives

$$\left(\begin{array}{c|ccc} h_{22} + \cdots + h_{nn} & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & h_{\alpha\beta} & \\ 0 & & & \end{array}\right)$$

where $2 \leq \alpha, \beta \leq n$. We see immediately that there is an n-dimensional kernel (we can let h_{1k} equal anything we want and if everything else is zero, we are in the kernel).

Now we will see how to overcome this issue. Rewrite the linearization as

$$\begin{aligned} \dot{R}_{jk} &= -\frac{1}{2} g^{im} \nabla_i \nabla_m \dot{g}_{jk} - \frac{1}{2} g^{im} \nabla_j \nabla_k \dot{g}_{im} + \frac{1}{2} g^{im} (\nabla_k \nabla_i \dot{g}_{jm} + \nabla_j \nabla_m \dot{g}_{ik}) \\ &= -\frac{1}{2} g^{im} \nabla_i \nabla_m \dot{g}_{jk} + \frac{1}{2} \nabla_k V_j + \nabla_j V_k \end{aligned}$$

if

$$V_j = g^{im} \nabla_i \dot{g}_{jm} - \frac{1}{2} \nabla_j \left(g^{im} \dot{g}_{im} \right).$$

The last term is equal to the Lie derivative $L_V g_{jk}$ (where $V = V^i = g^{ij} V_j$, often denoted $V^{\#}$) and so we get that the linearization of $-2R_{jk}$ is

$$g^{im} \nabla_i \nabla_m \dot{g}_{jk} - L_V g_{jk}.$$

Lie derivatives arise from changing by diffeomorphisms, i.e., if ϕ_t are diffeomorphisms such that

$$\frac{d}{dt} \phi_t (x) = X (x)$$

and ϕ_0 is the identity (i.e., ϕ_t is the flow of X), then

$$\left. \frac{d}{dt} \right|_{t=0} \phi_t^* g_0 = L_X g_0.$$

One can pretty easily see that if we take the vector field V as above, we can look at the flow ϕ_t and $\phi_t^* g(t)$ and we will see that $\tilde{g}(t) = \phi_t^* g(t)$ evolves by

$$\frac{\partial}{\partial t} \tilde{g} = -2 \operatorname{Rc} (\tilde{g}) + L_V \tilde{g}$$

and the linearization is

$$g^{im} \nabla_i \nabla_m \dot{g}_{jk}.$$

This is like looking at an equation roughly like

$$\frac{\partial}{\partial t} h = g^{im} \nabla_i \nabla_m h_{jk},$$

which is a heat equation with a unique solution. This has principal symbol g^{ij}, which is strictly positive definite. It can be shown that this implies that the modified Ricci flow (the equation above on $\tilde{g}$) has a unique solution. One can then show that this implies the Ricci flow has a unique solution too. I.e.,

Theorem 7 *Given an initial closed Riemannian manifold* (M, g_0), *there is a time* $T > 0$ *and Riemannian metrics* $g(t)$ *on* M *for each* $t \in [0, T)$ *such that which satisfy the initial value problem*

$$\begin{aligned} \frac{\partial}{\partial t} g &= -2 \operatorname{Rc} (g) \\ g(0) &= g_0. \end{aligned}$$

Moreover, given the initial condition, $g(t)$ *are uniquely determined, and there is a maximal such* T.

Remark 8 *One can also show that this is true for complete manifolds with bounded curvature* $|\mathrm{Rm}|$, *which was done by Shi. However, the proof is much more difficult on noncompact manifolds.*