Singularities of Ricci flow, limits, and κ-noncollapse

David Glickenstein
Math 538, Spring 2009

February 18, 2009

1 Introduction

This roughly covers Lecture 7 of Tao. We will look at analysis of singularities of Ricci flow.

2 Finite time singularities

Recall that we proved short time existence for Ricci flow, which means the flow exists until it reaches a singularity. The first, most basic result on singularities is the following result of Hamilton:

Theorem 1 *Let $(M, g(t))$ be a solution to the Ricci flow on a compact manifold on a maximal time interval $[0,T)$. If $T < \infty$, then*

$$\lim_{t \to T^-} \max_{x \in M} |\mathrm{Rm}(t)|^2_{g(t)} = \infty.$$

Remark 2 *Uniqueness of the Ricci flow is necessary for there to exist a maximal time interval. The idea is that if there is a solution on $[0,T_1)$ and another solution on $[T_1 - \varepsilon, T_2)$, then they must agree on the overlap, so one can consider the flow on $[0,T_2)$ which extends both the flows. Now simply take the sup of all T_2 for which the flow exists and this forms the maximal time interval.*

Remark 3 *N. Sesum was able to replace $|\mathrm{Rm}|$ with $|\mathrm{Rc}|$.*

Proof (idea). Suppose $T < \infty$ and $|\mathrm{Rm}|$ remains bounded. Then one can show that all derivatives $\left|\nabla^k \mathrm{Rm}\right|$ are uniformly bounded for $t \in [\varepsilon, T)$. One can use this to derive uniform bounds on the metric and its derivatives and then to extract a smooth limit metric at T (of a subsequence using an Arzela-Ascoli type compactness theorem). The existence/uniqueness result tells us that we can extend the flow for a short time, contradicting the fact that $[0,T)$ was maximal. ■

This is quite a useful theorem, but it is still possible to develop singularities in a finite time (for instance, the sphere or a neck pinch). It will be very important to analyze what is happening at the singular times so that we can do surgery to remove the problems and then continue the flow. We will do this using a PDE technique called blowing up around the singularity, which uses scaling to allow us to see the precise behavior of the flow near the singular time.

3 Blow ups

Here is the idea. At the singular time, we know that $|\mathrm{Rm}|^2$ is going to infinity. If we scale the metric g to cg, then we get

$$|\mathrm{Rm}\,(cg)|^2 = \frac{1}{c^2}\,|\mathrm{Rm}\,(g)|^2$$

or

$$|\mathrm{Rm}\,(cg)| = \frac{1}{c}\,|\mathrm{Rm}\,(g)|\,,$$

so if we want to prevent $|\mathrm{Rm}|$ from going to infinity, we choose a scaling

$$L_n^{-2} = \max_{x \in M} |\mathrm{Rm}\,(g\,(t_n))|$$

and rescale the metric $g\,(t_n)$ by L_n^{-2}. We can actually rescale to a sequence of solutions of the differential equation (Ricci flow) by looking at

$$g_n\,(t) = \frac{1}{L_n^2}\,g\left(t_n + tL_n^2\right),$$

where $t_n \to T$, the singular time. Notice that

$$\begin{aligned} \frac{\partial}{\partial t} g_n &= \frac{\partial}{\partial t}\left[L_n^{-2} g\left(t_n + tL_n^2\right)\right] \\ &= -2\,\mathrm{Rc}\left[g\left(t_n + tL_n^2\right)\right] \\ &= -2\,\mathrm{Rc}\left[g_n\,(t)\right] \end{aligned}$$

so g_n is a sequence of solutions to the Ricci flow whose initial value is getting closer to the singular time. Furthermore, the initial curvatures $|\mathrm{Rm}\,(g_n\,(0))|$ are all bounded by 1. On the down side, $L_n \to 0$ and so the metric is being multiplied by larger and larger scaling factors and thus it is quite likely that a limit will become noncompact. We will need to have a good notion of convergence which allows convergence to noncompact manifolds.

Remark 4 *We do not necessarily have to choose L_n as above. The fact that $L_n^{-2} \to \infty$ is why this is called a* blow-up. *If we take $L_n^{-2} \to 0$ then we have what is called a* blow-down.

Notice that if the original Ricci flow is defined on an interval $[0, T)$, then the rescaled solutions $g_n(t)$ exist on the interval

$$\left[-\frac{t_n}{L_n^2}, \frac{T - t_n}{L_n^2}\right).$$

Thus if we can extract a limit and $t_n \to T$ and $L_n \to 0$, then the limit metrics will be ancient, i.e., will start at $t = -\infty$ (the final endpoint depends a little more on how we choose the t_n with respect to how we pick the L_n, allowing it to be 0, positive, or $+\infty$; we may have reason to choose any of these).

The main goal is to find quantities which

1. become better as we go to the limit (Tao calls these critical or subcritical) and
2. severely restrict the geometry of the limit (Tao calls these coercive).

Examples of quantities to study or not:

- The volume of the manifold. If we look at

  $$V(M, g) = \int_M dV_g,$$

  we see that

  $$V\left(M, \frac{1}{L_n^2} g\right) = \frac{1}{L_n^{2/d}} V(M, g)$$

  and so if $L_n \to 0$, this quantity does not persist to the limit. Tao calls this behavior supercritical.

- The total scalar curvature. If we look at

  $$F(M, g) = \int_M R dV,$$

  we see that

  $$F\left(M, \frac{1}{L_n^2} g\right) = L_n^{2-d} F(M, g).$$

  Thus it is preserved in dimension 2 (this is the Gauss-Bonnet theorem, which implies critical behavior) and does not persist in higher dimensions (supercritical).

- The minimum scalar curvature. If we look at

  $$R_{\min}(M, g) = \min_{x \in M} R(x),$$

  then we see that

  $$R_{\min}\left(M, \frac{1}{L_n^2} g\right) = L_n^2 R_{\min}(M, g)$$

  and so this goes to zero as $L_n \to 0$. This is subcritical behavior. Unfortunately, $R_{\min} = 0$ does not give sufficient information of the limit to classify (not coercive enough).

- Lowest eigenvalue of the operator $-4\triangle u+R$. Notice that this is subcritical since if we call this functional $H(M,g)$

$$H\left(M,\frac{1}{L_n^2}g\right)=L^2H(M,g).$$

  It turns out that this quantity obeys a monotonicity and has some nice coercivity, though not quite enough. We will soon look at a particular critical (i.e., scale invariant) quantity which is related.

4 Convergence and collapsing

Manifolds may converge in a number of ways. Here are some examples:

- Sphere converging to a point
- Sphere converging to a cylinder
- Cylinder collapsing to a line
- Torus collapsing to a circle
- Torus collapsing to a line
- 3-sphere collapsing to a 2-sphere by shrinking Hopf fibers

There are several issues here, notably:

- Is there collapse?
- Does convergence involve noncompact manifolds?

The most obvious notion of collapse involves the injectivity radius going to zero. Recall that the exponential map is the map from the tangent space at one point to the manifold where a vector v is taken to the point one unit from the origin along a geodesic starting with velocity v. This map is a local diffeomorphism, and there is an $r>0$ such that the ball $B(0,r)$ is mapped diffeomorphically to a ball on the manifold. The largest such r is called the *injectivity radius*. As this goes to zero, there is collapse.

We will see another way to measure this collapse soon.

For noncompact manifolds, one needs to consider pointed convergence. This generally involves looking at convergence of balls of larger and larger size. If all manifolds and their limit have a uniform diameter bound, then one does not need to consider pointed convergence.

5 κ-noncollapse

One definition of a collapsing sequence is the following:

Definition 5 *A pointed sequence (M_n, g_n, p_n) of Riemannian manifolds is collapsing if* $\mathrm{inj}_{p_n} \to 0$ *as* $n \to \infty$.

We may wish to rescale the manifolds (M_n, g_n) to be of some uniform size, say by making $|\mathrm{Rm}(p_n)| = 1$. Then one can consider a rescaled collapsing if $|\mathrm{Rm}(p_n)|^{1/2} \mathrm{inj}_{p_n} \to 0$.

When one assumes that the sectional curvature is bounded, then the collapse is restricted. Let $V(U) = V_g(U)$ denote the Riemannian volume of the Borel subset $U \subset M$.

Theorem 6 (Cheeger) *Suppose that* $|\mathrm{Rm}|_g \leq C r_0^{-2}$ *on* $B(p, r_0) \subset M^d$ *and that*

$$V(B(p, r_0)) \geq \delta r_0^d$$

for some $\delta > 0$. *Then the injectivity radius of* p, inj_p, *is at least*

$$\mathrm{inj}_p \geq c r_0$$

for some constant $c = c(C, \delta, d) > 0$.

Let's think about this theorem for a minute, to see if it is ever applicable since it seems the assumptions are quite strong. Note that as $r_0 \to 0$, the ball looks more and more Euclidean. That means that for very small $r_0 \ll \mathrm{inj}_p$,

$$V(B(p, r_0)) \approx \omega r^d,$$

where $\omega = \omega(d)$ is the correct constant for a Euclidean ball, and

$$|\mathrm{Rm}|_g \approx |\mathrm{Rm}(p)|_g .$$

So as $r_0 \to 0$, we see that

$$\lim_{r_0 \to 0} \left(r_0^2 \sup_{x \in B(p, r_0)} |\mathrm{Rm}(x)| \right) = 0$$

$$\lim_{r_0 \to 0} \frac{V(B(p, r_0))}{\omega r^d} = 1.$$

In particular, for any $C > 0$ and $0 < \delta < 1$, there is a $r_* > 0$ such that the assumptions are satisfied if $r_0 \leq r_*$.

Note that the converse is also true from more classical results.

Theorem 7 *If* $|\mathrm{Rm}|_g \leq C$ *and* $\mathrm{inj}_p \geq \iota$, *then there is a* $\delta = \delta(C, \iota, d)$ *such that*

$$V(B(p, r)) \geq \delta r^d$$

for all $r \leq \iota$.

Collapse generally refers to the injectivity radius going to zero. Cheeger's theorem tells us that when curvature is bounded, volume of balls getting small and injectivity radius getting small are essentially the same. This roughly motivates the following noncollapsing definition, which is not quite the definition we will use.

Definition 8 *A Riemannian manifold* $\left(M^d, g\right)$ *is* κ*-collapsed at* $p \in M$ *at scale* r_0 *if*

1. *(Bounded normalized curvature)* $|\mathrm{Rm}|_g \leq r_0^{-2}$ *for all* $x \in B(p, r_0)$ *and*
2. *(Volume collapsed)* $V(B(p, r_0)) \leq \kappa r_0^d$.

If these are not satisfied, then we say the manifold is κ-noncollapsed at p at the scale r_0. Is this a reasonable definition? Here are some observations:

- By Cheeger's theorem, this would imply a lower bound on injectivity radius.
- Note that if the one is on a ball where the curvature is large, then the manifold is automatically κ-noncollapsed at that scale.
- By the discussion above, every manifold is κ-noncollapsed at a small enough scale and κ smaller than the constant for a Euclidean ball.
- This definition is scale independent in the following sense. If we consider $\bar{g} = r_0^{-2} g$, then the conditions are

$$\begin{aligned} |\mathrm{Rm}(\bar{g})|_{\bar{g}} &\leq 1 \\ V(B_{\bar{g}}(p, 1)) &\leq \kappa. \end{aligned}$$

- The sphere $\mathbb{S}^n$ is κ-noncollapsed at scales r_0 less than the diameter (for a suitable choice of κ). The bounded normalized curvature assumption of the definition are satisfied for $r_0 \leq 1$, although one might argue that there is really no local collapsing at scales less than π. Apparently this will not be important for our argument. At large scales, the curvature assumption fails.
- The flat torus has $|\mathrm{Rm}| = 0$, and so it is noncollapsed at scales less than the injectivity radius. The curvature assumption is valid for large scales, but for a given κ, if r_0 is taken large enough, the torus must be collapsed, since the volume is never larger than the volume of the torus.
- We want to consider whether there exists a κ such that a manifold is κ-noncollapsed at large and small scales. Certainly, one can make κ small enough (say, less than the constant for the area of a Euclidean ball) so that a manifold is κ-collapsed at many scales, but this is not of use to us.

Perelman adapted these ideas to Ricci flow (time dependent metrics) as follows.

Definition 9 *Let $\left(M^d, g(t)\right)$ be a solution to Ricci flow and let $\kappa > 0$. Then Ricci flow is κ-collapsed at a point (t_0, x_0) in spacetime at a scale r_0 if:*

1. *(Bounded normalized curvature)*
$$|\mathrm{Rm}(t,x)|_{g(t)} \leq r_0^{-2}$$
for all $(t,x) \in \left[t_0 - r_0^2, t_0\right] \times B_{g(t_0)}(x_0, r_0)$, and

2. *(Collapsed volume)*
$$V\left(B_{g(t_0)}(x_0, r_0)\right) \leq \kappa r_0^d.$$

Otherwise, we say that the solution is κ-noncollapsed at p at scale r_0.

Remark 10 *Notice that the assumptions require Ricci flow to exist on the time interval $[t_0 - r_0^2, t_0]$.*

Remark 11 *As remarked above, for each κ smaller than the volume of a unit ball in Euclidean space, there is a r_* so that M is κ-noncollapsed at scales less than r_*. For Ricci flow, one can still find r_*, but it will, in general depend on t_0. As t_0 goes to a singular time, it may be possible that $r_* \to 0$. This is what we would like to rule out, as we shall see.*

Remark 12 *Notice that κ is dimensionless.*

Remark 13 *If $g(t)$ is κ-noncollapsed at (t_0, x_0) at the scale r_0, we see that $Kg\left(t_* + \frac{t}{K}\right)$ is κ-noncollapsed at $(K(t_0 - t_*), x_0)$ at the scale of $K^{-1/2} r_0$.*

Here is a typical noncollapsing theorem along the lines of Perelman.

Theorem 14 (Perelman's noncollapsing theorem, first version) *Let $(M, g(t))$ be a solution to the Ricci flow on compact 3-manifolds for $t \in [0, T_0]$ such that at $t = 0$ we have*

$$|\mathrm{Rm}(p)|_{g(0)} \leq 1$$
$$V\left(B_{g(0)}(p,1)\right) \geq \omega$$

for all $p \in M$ and $\omega > 0$ fixed. Then there exists $\kappa = \kappa(\omega, T_0) > 0$ such that the Ricci flow is κ-noncollapsed for all $(t_0, x_0) \in [0, T_0] \times M$ and scales $0 < r_0 < \sqrt{t_0}$.

A big point of this theorem is that it rules out a limit of $\Sigma \times \mathbb{R}$ where Σ is the cigar soliton solution of Hamilton. The manifold $\Sigma \times \mathbb{R}$ is essentially a fixed point of Ricci flow (when we consider it a flow on metric spaces, not Riemannian metrics). Σ is a positive curvature metric on $\mathbb{R}^2$ which has maximum curvature at the origin and is asymptotic to a cylinder as one moves away from the origin.

This implies that $\Sigma \times \mathbb{R}$ has volume $V(B(0,r))$ asymptotic to Cr^2 for large r (For a cylinder, notice that large balls of radius r have volumes asymptotic to Cr, not Cr^2.) Thus, $\Sigma \times \mathbb{R}$ is not κ-noncollapsed at large scales (r_0 large).

Consider the blowups $(M, g_n(t))$ defined above. Note that we have a κ such that the Ricci flow $g(t)$ is κ-noncollapsed for time interval $[0, T_0]$ for any $T_0 < T$. Thus, by Remark 13 we must have that $g_n(t)$ is κ-noncollapsed at the scale of $L_n^{-1} r_0$ for the time interval $\left[-\frac{t_n}{L_n^2}, \frac{T-T_0}{L_n^2}\right)$. As n becomes large, we see that $g_n(t)$ becomes κ-noncollapsed at all scales, which is not true for $\Sigma \times \mathbb{R}$.

We will describe this in more detail later in the course. We will now move to proving this theorem, the subject of the next few lectures.