Perelman entropy and κ-noncollapse

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1 Gradient flow

Formulating an equation as a gradient flow has many advantages. Consider the heat equation

$$\frac{\partial f}{\partial t} = \triangle f$$

on a compact Riemannian manifold. It is easy to see that if one considers the energy

$$E(f) = \frac{1}{2}\int_M |\nabla f|_g \, dV,$$

that if we take the time derivative of the energy when f satisfies the heat equation, we get

$$\begin{aligned} \frac{dE}{dt}(f) &= \int_M \nabla f \cdot \nabla\left(\frac{\partial}{\partial t} f\right) dV \\ &= -\int_M \triangle f \left(\frac{\partial}{\partial t} f\right) dV \\ &= -\int_M (\triangle f)^2 \, dV \leq 0. \end{aligned}$$

Thus we immediately get that the energy is decreasing and that stationary points are harmonic functions, i.e., functions which satisfy $\triangle f = 0$. This monotonicity also tells us that f cannot have periodic solutions which are not fixed points, for if $f(t_1, x) = f(t_2, x)$ for all x, then $E(f(t_1, \cdot)) = E(f(t_2, \cdot))$, and the monotonicity implies that $\triangle f = 0$ for $t \in [t_1, t_2]$.

The monotonicity is true in general for a gradient flow. If one has an energy $E(f)$, one defined the gradient flow as

$$\frac{\partial f}{\partial t} = \pm grad(E),$$

where the gradient vector $grad(E)$ is given so that

$$dE(X) = g(X, grad(E)),$$

for some metric g on the space of functions. In our case that g is the L^2 metric (which uses the Riemannian metric g on M).

It would be nice to represent Ricci flow in this way. It is not at all trivial to do this.

2 Ricci flow as a gradient flow

An obvious choice of functional is the Einstein-Hilbert functional:

$$EH(g) = \int_M R dV.$$

To calculate its variation, recall that if we have a variation of the metric $(\delta g)_{ij} = h_{ij}$, then we get

$$\delta R(h) = -\langle \mathrm{Rc}, h\rangle - \triangle \operatorname{tr}_g(h) + \operatorname{div}\operatorname{div} h.$$

It is not hard to see that since

$$\delta \log\det g = g^{ij} h_{ij}$$

so

$$\begin{aligned}\delta\sqrt{\det g} &= \delta \exp\left(\frac{1}{2}\log\det g\right)\\ &= \frac{1}{2}(\operatorname{tr}_g h)\sqrt{\det g}\end{aligned}$$

so

$$\delta(dV) = \frac{1}{2}(\operatorname{tr}_g h)\, dV.$$

Using the above formula we get

$$\begin{aligned}\delta(EH)(h) &= \int_M \left(-\langle \mathrm{Rc}, h\rangle - \triangle \operatorname{tr}_g(h) + \operatorname{div}\operatorname{div} h + \frac{1}{2}R(\operatorname{tr}_g h)\right) dV\\ &= \int_M \left(-\langle \mathrm{Rc}, h\rangle + \frac{1}{2}R(\operatorname{tr}_g h)\right) dV\\ &= \int_M \left\langle \frac{1}{2}Rg - \mathrm{Rc}, h\right\rangle dV.\end{aligned}$$

Remark 1 *Here we used the divergence theorem for a compact Riemannian manifold, which says that*

$$\int \langle \operatorname{div} T, S\rangle\, dV = -\int \langle T, \nabla S\rangle\, dV,$$

for any n-tensor T and $(n-1)$-tensor S. More explicitly,

$$\int g^{i_1 j_1}\cdots g^{i_n j_n} g^{j i_0} \nabla_j T_{i_0 i_1\cdots i_n} S_{j_1 j_2\cdots j_n} dV = \int g^{i_1 j_1}\cdots g^{i_n j_n} g^{j i_0} T_{i_0 i_1\cdots i_n} \nabla_j S_{j_1 j_2\cdots j_n} dV.$$

So critical points of the Einstein-Hilbert functional satisfy the Einstein equation. The gradient flow would be

$$\frac{\partial}{\partial t}g = -2\left(\mathrm{Rc} - \frac{1}{2}Rg\right).$$

The problem is that this flow is not parabolic and there is no existence theory for such equations.

Let's try a new tactic. Replace dV by a fixed measure dm. Then the functional

$$H(g) = \int_M R\,dm$$

satisfies the variation

$$\delta H(h) = \int_M \left(-\langle \mathrm{Rc}, h\rangle - \triangle \operatorname{tr}_g(h) + \operatorname{div}\operatorname{div} h\right) dm.$$

You don't lose the last two terms with the divergence theorem, since that only works with the volume measure. However, we can consider the Radon-Nikodym derivative and write

$$dm = \frac{dm}{dV}dV$$

for a positive function $\frac{dm}{dV}$. We can write

$$\frac{dm}{dV} = e^{-f}.$$

Then,

$$\delta H(h) = \int_M \left(-\langle \mathrm{Rc}, h\rangle - \triangle \operatorname{tr}_g(h) + \operatorname{div}\operatorname{div} h\right) e^{-f}dV$$

can be integrated by parts to get

$$\begin{aligned}
\delta H(h) &= \int_M \left(-\langle \mathrm{Rc}, h\rangle - \langle \nabla \operatorname{tr}_g(h), \nabla f\rangle + \operatorname{div} h \cdot \nabla f\right) e^{-f}dV \\
&= \int_M \left(-\langle \mathrm{Rc}, h\rangle + \operatorname{tr}_g(h)\triangle f - \operatorname{tr}_g(h)|\nabla f|^2 - \left\langle h, \nabla^2 f\right\rangle + h(\nabla f, \nabla f)\right) e^{-f}dV \\
&= \int_M \left\langle -\mathrm{Rc} - \nabla^2 f + \left(\triangle f - |\nabla f|^2\right) g + \nabla f \nabla f, h\right\rangle e^{-f}dV.
\end{aligned}$$

Remark 2 *Tao often uses $\langle\cdot,\cdot\rangle$ to denote the Euclidean metric locally, and so explicitly puts in g^{ij}'s in this case. We will understand that $\langle\cdot,\cdot\rangle$ requires the metric, and so when quantities like this are differentiated, we also need to differentiate the metric, as we will see below.*

We need to add another term, and so we get

$$F(g) = \int_M \left(R + |\nabla f|^2\right) e^{-f}dV = \int_M \left(R + \left|\nabla \log \frac{dm}{dV}\right|^2\right) dm.$$

We now need that

$$
\begin{aligned}
0 = \delta\left(dm\right) = \delta\left(e^{-f}dV\right) &= -\left(\delta f\right)e^{-f}dV + \frac{1}{2}\operatorname{tr}_g he^{-f}dV \\
&= \left(-\delta f + \frac{1}{2}\operatorname{tr}_g h\right)e^{-f}dV
\end{aligned}
$$

Thus we have that

$$\delta f = \frac{1}{2}\operatorname{tr}_g h$$

Let

$$E\left(g\right) = \int\left(g^{ij}\nabla_i f\nabla_j f\right)e^{-f}dV$$

where $dm = e^{-f}dV$ is fixed (so that $f = -\log dm/dV$ and δf is expressed as above). We can then compute

$$
\begin{aligned}
\delta E\left(h\right) &= \delta\left[\int\left(g^{ij}\nabla_i f\nabla_j f\right)e^{-f}dV\right] \\
&= \int\left(-\left\langle h,\nabla f\nabla f\right\rangle + 2\left\langle\nabla f,\nabla\left(\delta f\right)\right\rangle - \left|\nabla f\right|^2\delta f + \frac{1}{2}\left|\nabla f\right|^2\operatorname{tr}_g h\right)e^{-f}dV \\
&= \int\left(-\left\langle h,\nabla f\nabla f\right\rangle - 2\triangle f\left(\delta f\right) + \left|\nabla f\right|^2\delta f + \frac{1}{2}\left|\nabla f\right|^2\operatorname{tr}_g h\right)e^{-f}dV \\
&= \int\left(\left\langle h,-\nabla f\nabla f - \left(\triangle f\right)g + \left|\nabla f\right|^2 g\right\rangle\right)e^{-f}dV.
\end{aligned}
$$

Now, since $F = H + E$, we have

$$\delta F\left(h\right) = \int_M\left\langle -\operatorname{Rc} - \nabla^2 f, h\right\rangle e^{-f}dV.$$

Thus the gradient flow of $-2F$ is

$$\frac{\partial g}{\partial t} = -2\operatorname{Rc}\left(g\right) - 2\nabla^2 f.$$

This is almost Ricci flow, but not quite. The f is changing, too, by the equation

$$\frac{\partial f}{\partial t} = -\triangle f - R.$$

This is a backward heat equation, which it turns out will make it useful to probe backwards.

Notice that

$$2\nabla^2 f = \mathcal{L}_{\nabla f}g,$$

the Lie derivative of g in the direction ∇f. This means that the flow above differs from Ricci flow by a diffeomorphism. Instead, we can consider $\bar{g} = \phi^* g$, where ϕ is the flow of diffeomorphisms generated by ∇f, and we will see that $\bar{g}$ evolves by Ricci flow. Furthermore, f will differ by a Lie derivative, and

$$\mathcal{L}_{\nabla f}f = df\left(\nabla f\right) = \left|\nabla f\right|^2$$

(where is the metric in here? It is in ∇f, which is a vector field gotten by raising the index on df) and so under the new flow, $\bar{f} = f \circ \phi$ evolves by

$$\frac{\partial \bar{f}}{\partial t} = -\triangle_{\bar{g}} \bar{f} - \bar{R} + \left|\nabla \bar{f}\right|_{\bar{g}}^{2}.$$

Example 3 (Fundamental, important example) *Let (M, g) be Euclidean space $M = \mathbb{R}^d$ and let*

$$f(t, x) = \frac{|x|^2}{4\tau} + \frac{d}{2}\log 4\pi\tau = -\log\left[(4\pi\tau)^{-d/2} e^{-|x|^2/(4\tau)}\right]$$

where $\tau = t_0 - t$, Notice that $e^{-f}dx$ is the Gaussian measure, which solves the backward heat equation (the fundamental solution to the heat equation). If $t < t_0$, this choice of g and f satisfy the equations

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Rc}(g) \tag{1}$$

$$\frac{\partial f}{\partial t} = -\triangle f - R + |\nabla f|^2. \tag{2}$$

We can check:

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial}{\partial t}\left[\frac{|x|^2}{4\tau} + \frac{d}{2}\log 4\pi\tau\right] \\ &= \frac{|x|^2}{4\tau^2} - \frac{d}{2\tau}.\end{aligned}$$

$$\begin{aligned}\nabla f &= \frac{x}{2\tau} \\ |\nabla f|^2 &= \frac{|x|^2}{4\tau^2} \\ \triangle f &= \frac{d}{2\tau}\end{aligned}$$

so it works.

Notice that when we pull back by ϕ, the measure $\phi^* dm$ is not static. Thus it makes sense to rewrite the functional as

$$F(M, g, f) = \int_M \left(R + |\nabla f|^2\right) e^{-f} dV.$$

Notice that this functional is invariant under diffeomorphism, i.e.,

$$F(M, \phi^* g, f \circ \phi) = F(M, g, \phi)$$

for any diffeomorphisms ϕ. We also have that under the coupled flows (1) and (2),

$$\frac{\partial F}{\partial t}(M, g, f) = 2\int \left|\mathrm{Rc} + \nabla^2 f\right|^2 e^{-f} dV.$$

Thus we have that F is monotone increasing under Ricci flow. Unfortunately, now we have to explicitly deal with this quantity f. To eliminate this, we take the infimum:

$$\lambda(M,g) = \inf\left\{F(M,g,f) : \int_M e^{-f} dV = 1\right\}$$

(the infimum is over f). Using the following exercise, we can show that λ is finite.

Exercise 4 *Show that $\lambda(M,g)$ is the smallest number for which one has the inequality*

$$\int_M \left(4|\nabla u|_g^2 + Ru^2\right) dV \geq \lambda \int_M u^2 dV,$$

where u is in $H^1(M) = W^{1,2}(M)$, the Sobolev space of functions with 1 derivative in L^2 (so it has norm $\|f\|_{H^1} = \int\left(|\nabla f|_g^2 + f^2\right) dV$ for C^1 functions). Hint: show that we can assume u is positive and then write $u = e^{-f/2}$. Thus λ is the smallest eigenvalue of the operator $-4\triangle_g u + R$.

Using the exercise, one sees that the fact that every compact manifold satisfies a Poincaré inequality,

$$\int_M |\nabla u|^2 dV \geq c(d) \int_M u^2 dV,$$

implies that λ is bounded below, basically, by the best constant in the Poincaré inequality plus $\min R$. Note that the Poincaré inequality constant depends on the dimension. We will see later a similar inequality for which the constant does not depend on dimension.

Furthermore, one can prove that λ is realized by a positive function $u = e^{-f/2}$ with $\|u\|_{L^2(M)} = 1$. Note that $H^1(M)$ embeds compactly into $L^2(M)$ since M is compact. Thus if we take a minimizing sequence $\{u_n\}$ in H^1, there is a subsequence (which we also denote by $\{u_n\}$ which converges in L^2 to a function u. Now consider:

$$\begin{aligned}
&\int \left(|\nabla u_n|^2 + Ru_n^2\right) dV + \int \left(|\nabla u_m|^2 + Ru_m^2\right) dV \\
&= \frac{1}{2}\int \left(|\nabla(u_n - u_m)|^2 + R(u_n - u_m)^2\right) dV + \frac{1}{2}\int \left(|\nabla(u_n + u_m)|^2 + R(u_n + u_m)^2\right) dV
\end{aligned}$$

so

$$\begin{aligned}
\frac{1}{2}\int \left(|\nabla(u_n - u_m)|^2 + R(u_n - u_m)^2\right) dV &= \int \left(|\nabla u_n|^2 + Ru_n^2\right) dV + \int \left(|\nabla u_m|^2 + Ru_m^2\right) dV \\
&\quad - \frac{1}{2}\int \left(|\nabla(u_n + u_m)|^2 + R(u_n + u_m)^2\right) dV.
\end{aligned}$$

The right side goes to zero in the limit since the terms go to λ, λ and -2λ. Also, we know that

$$-\min|R|\,\|u_n - u_m\|_{L_2} \leq \int R\,(u_n - u_m)^2\,dV \leq \min|R|\,\|u_n - u_m\|_{L_2}$$

and each term goes to zero. Thus we know that the sequence $\{u_n\}$ is Cauchy in H^1 and since H^1 is complete, it must converge to a function in H^1.

Since λ is attained at a function, we can sometimes prove inequalities like the following (taken from the notes of Kleiner and Lott):

Let $h\,(s,t,x)$ be a two-parameter family of functions such that

$$\begin{aligned} h\,(s,t_0) &= f_*\,(t_0+s) \\ \frac{\partial h}{\partial t} &= -\triangle h + -R + |\nabla h|^2\,. \end{aligned}$$

There is a solution to this for $t \leq t_0$ since the equation is backwards elliptic. (Note: we have only shown that f_* is in H^1, so we mean a weak solution to the parabolic equation. We could also show that f_*, as a minimizer, satisfies a particular elliptic equation, which implies that f_* is smooth by elliptic regularity theory.)

$$\begin{aligned} \lambda\,(t_0) &\leq F\,(M, g\,(t_0)\,, h\,(s, t_0 - s)) \\ &= F\,(M, g\,(t_0+s)\,, h\,(s,t_0)) - 2\int_0^s \left[\int_M \left|\mathrm{Rc}\,(g\,(t_0+\sigma)) + \nabla^2 h\,(s, t_0+\sigma)\right|^2 e^{-h(s,t_0+\sigma)}dV\right] d\sigma. \end{aligned}$$

We then have

$$\lambda\,(t_0) \leq \lambda\,(t_0+s) - 2\int_0^s \left[\int_M \left|\mathrm{Rc}\,(g\,(t_0+\sigma)) + \nabla^2 h\,(s, t_0+\sigma)\right|^2 e^{-h(s,t_0+\sigma)}dV\right] d\sigma$$

and so

$$\begin{aligned} \frac{\partial\lambda}{\partial t}\,(t_0) &= \lim_{s\to 0+} \frac{\lambda\,(t_0+s) - \lambda\,(t_0)}{s} \\ &\geq \lim_{s\to 0+} \frac{2}{s}\int_0^s \left[\int_M \left|\mathrm{Rc}\,(g\,(t_0+\sigma)) + \nabla^2 h\,(s, t_0+\sigma)\right|^2 e^{-h(s,t_0+\sigma)}dV\right] d\sigma \\ &= 2\int_M \left|\mathrm{Rc}\,(g\,(t_0)) + \nabla^2 h\,(0,t_0)\right|^2 e^{-h(0,t_0)}dV \\ &= 2\int_M \left|\mathrm{Rc}\,(g\,(t_0)) + \nabla^2 f_*\,(t_0)\right|^2 e^{-f_*}dV \end{aligned}$$

We can now derive

$$
\begin{aligned}
\frac{\partial \lambda}{\partial t} &\geq 2\int \left|\operatorname{Rc} + \nabla^2 f_*\right|^2 e^{-f_*} dV \\
&\geq \frac{2}{3}\int \left(R + \triangle f_*\right)^2 e^{-f_*} dV \\
&\geq \frac{2}{3}\left[\int \left(R + \triangle f_*\right) e^{-f_*} dV\right]^2 \\
&= \frac{2}{3}\left[\int \left(R + \left|\nabla f_*\right|^2\right) e^{-f_*} dV\right]^2 \\
&= \frac{2}{3}\lambda^2.
\end{aligned}
$$

3 Perelman entropy

We now wish to make our functional scale invariant (so that we get a critical quantity, not just subcritical). In particular, we know that

$$
\frac{dF_m}{dt}\left(M, g\right) = 2\int \left|\operatorname{Rc} + \nabla^2 f\right|^2 e^{-f} dV
$$

is fixed (under the gradient flow) if

$$
\operatorname{Rc} = -\nabla^2 f,
$$

i.e., if (M, g) is a gradient Ricci soliton. We wish to have a new functional which is fixed if (M, g) is a gradient shrinking soliton, i.e.,

$$
\operatorname{Rc} = -\nabla^2 f + \frac{1}{2\tau} g
$$

for some $\tau > 0$. A round sphere is a gradient shrinking soliton, so it makes sense that we would want something like this. Under the Ricci flow, this structure is preserved except that τ decreases at a constant rate.

First note that if we consider the Nash entropy

$$
N_m\left(M, g\right) = \int_M \frac{dm}{dV} \log \frac{dm}{dV} dV = \int_M \left(\log \frac{dm}{dV}\right) dm = -\int f dm
$$

then

$$
\begin{aligned}
\frac{dN_m}{dt} &= -\int_M \left(-\triangle f - R\right) dm \\
&= \int_M \left(R + \left|\nabla f\right|^2\right) dm = F_m\left(M, g\right).
\end{aligned}
$$

This will come in handy. Now, suppose we want a quantity $W\left(M, g\right)$ such that

$$
\frac{dW}{dt} = \int_M \left|\operatorname{Rc} + \nabla^2 f - \frac{1}{2\tau} g\right|^2 dm.
$$

But in this case, would not have scale invariance for W, since t scales like distance squared, so the integrand should scale like distance squared. We will fix this by assuming

$$\frac{d\tau}{dt} = -1$$

and trying for

$$\frac{dW}{dt} = 2\tau \int_M \left| \operatorname{Rc} + \nabla^2 f - \frac{1}{2\tau} g \right|^2 dm. \tag{3}$$

Now, to find such a quantity, consider

$$\left| \operatorname{Rc} + \nabla^2 f - \frac{1}{2\tau} g \right|^2 = \left| \operatorname{Rc} + \nabla^2 f \right|^2 - \frac{1}{\tau}\left(R + \triangle f\right) + \frac{d}{4\tau^2}.$$

Thus we have that

$$\begin{aligned} 2\tau \int_M \left| \operatorname{Rc} + \nabla^2 f - \frac{1}{2\tau} g \right|^2 dm &= \tau \frac{dF_m}{dt} - 2F_m + \frac{d}{2\tau} \\ &= \frac{d}{dt}\left(\tau F_m - N_m - \frac{d}{2}\log\tau \right). \end{aligned}$$

This is what our W_m would be. However, as we did last time, we wanted to reparametrize so that

$$dm = e^{-f} dV$$

where dV is evolving according to Ricci flow evolution. This time, we will change

$$\tilde{f} = f - \frac{d}{2}\log\left(4\pi\tau\right)$$

so that

$$dm = e^{-f} dV = \left(4\pi\tau\right)^{-d/2} e^{-\tilde{f}} dV.$$

Remark 5 *This looks like the heat kernel for Euclidean space, which is why this particular normalization is given.*

Note that the preservation of dm implies that

$$\frac{d}{2\tau} - \frac{\partial \tilde{f}}{\partial t} + \frac{1}{2}\operatorname{tr} h = 0.$$

Under the gradient flow

$$\frac{\partial g}{\partial t} = -2\operatorname{Rc} - 2\nabla^2 f,$$

we have

$$\frac{\partial \tilde{f}}{\partial t} = \frac{d}{2\tau} - R - \triangle f.$$

Thus we get for

$$\begin{aligned} W_m(M,g,\tau) &= \tau F_m - N_m - \frac{d}{2}\log\tau \\ &= \int \left[\tau\left(R + \left|\nabla \tilde{f}\right|^2\right) + \tilde{f} - \frac{d}{2}\log(4\pi)\right](4\pi\tau)^{-d/2} e^{-\tilde{f}} dV \end{aligned}$$

Actually, we usually renormalize this to vanish in the Euclidean case, and so we can change the d term appropriately to

$$W_m(M,g,\tau) = \int \left[\tau\left(R + \left|\nabla \tilde{f}\right|^2\right) + \tilde{f} - d\right](4\pi\tau)^{-d/2} e^{-\tilde{f}} dV.$$

We can also define the Perelman entropy as the functional

$$W(M,g,f,\tau) = \int \left[\tau\left(R + |\nabla f|^2\right) + f - d\right](4\pi\tau)^{-d/2} e^{-f} dV$$

where g is a Riemannian metric, f is a function on M, and τ is a positive constant.

If g satisfies the Ricci flow, then we need to pull back f to get the three evolutions

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Rc} \tag{4}$$

$$\frac{\partial f}{\partial t} = \frac{d}{2\tau} - R - \triangle f + |\nabla f|^2 \tag{5}$$

$$\frac{d\tau}{dt} = -1.$$

Under these three, the Perelman Entropy satisfies

$$\frac{dW}{dt}(M,g,f,\tau) = 2\tau \int_M \left|\mathrm{Rc} + \nabla^2 f - \frac{1}{2\tau} g\right|^2 (4\pi\tau)^{-d/2} e^{-f} dV.$$

We would like to find a minimum over all functions f and constants τ so that we have an invariant of the Riemannian manifold. However, it is not yet clear that such an infimum exists. Recall that last time, the existence followed from a Poincaré inequality. In this scale invariant setting, the existence of a minimizer will follow from a log-Sobolev inequality.

4 Log-Sobolev inequalities

Let's first consider what happens if g is the Euclidean metric. We would like to switch to a function which looks like the heat kernel, namely,

$$u = (4\pi\tau)^{-d/2} e^{-f}.$$

Recall that our model case is when $f = |x|^2 / (4\tau)$, in which case this is precisely the backwards heat kernel. The backwards heat kernel satisfies:

$$\frac{\partial u}{\partial t} = -\triangle u$$

for $\tau > 0$ and

$$\lim_{\tau \to 0^-} u(\tau, x) = \delta_0(x),$$

weakly, where δ_0 is the delta function. Furthermore, one can check that

$$\int_{\mathbb{R}^d} u dx = \int_{\mathbb{R}^d} (4\pi\tau)^{-d/2} e^{-|x|^2/(4\tau)} dx = 1 \tag{6}$$

for any τ. The backwards heat kernel can be used to solve the heat equation with some given final conditions, e.g., to solve

$$\frac{du}{dt} = -\triangle u$$
$$u(T, x) = f(x),$$

we see that the convolution

$$u(t, x) = \int_{\mathbb{R}^d} f(y) (4\pi\tau)^{-d/2} e^{-|x-y|^2/(4\tau)} dy$$

is a solution.

Exercise 6 *Show that all of this is true. Hint: to show (6), turn the integral into polar coordinates and assume the dimension is at least* 2. *For the dimension 1 case, there is a trick involving turning it into a dimension 2 integral and separating.*

We can check that for g Euclidean and f as above, we have

$$W(M, g, f, \tau) = \int \left[\frac{|x|^2}{2\tau} - d \right] (4\pi\tau)^{-d/2} e^{-|x|^2/(4\tau)} dx.$$

One can show that this is zero since

$$W(M, g, f, \tau) = \int \left[2\tau \left(|\nabla f|^2 - \triangle f \right) \right] (4\pi\tau)^{-d/2} e^{-f} dx,$$

and integrating by parts (needs to be justified) shows this is equal to zero.

Now we re-write W by replacing f with u. We see that (remembering still we are in Euclidean space),

$$W = \int \left[\tau \frac{|\nabla u|^2}{u^2} - u \log u \right] dx - \frac{d}{2} \log(4\pi\tau) - d$$

using identities such as

$$
\begin{aligned}
u &= (4\pi\tau)^{-d/2} e^{-f}. \\
\log u &= -\frac{d}{2}\log(4\pi\tau) - f \\
|\nabla u|^2 &= (4\pi\tau)^{-d} |\nabla f|^2 e^{-2f} \\
|\nabla f|^2 &= \frac{|\nabla u|^2}{u^2}.
\end{aligned}
$$

Tao shows that one can show that $W \geq 0$, which implies a log-Sobolev inequality

$$\tau \int \frac{|\nabla u|^2}{u^2} dx \geq \int u \log u dx + \frac{d}{2}\log(4\pi\tau) + d,$$

or as it is usually stated, with $\phi^2 = u$,

$$4\tau \int |\nabla \phi|^2 dx \geq \frac{1}{\tau}\int \phi^2 \log \phi^2 dx + \frac{d}{2}\log(4\pi\tau) + d.$$

For the general case, we have

$$W(M, g, f, \tau) = \int \left[\tau\left(Ru + \frac{|\nabla u|^2}{u^2}\right) - u \log u\right] dV - \frac{d}{2}\log(4\pi\tau) - d$$

One can show that the

$$W(M, g, f, \tau) \geq -C(M, g, \tau).$$

This implies essentially a log-Sobolev inequality, i.e.,

$$\tau \int R\phi^2 dV + \tau \int 4|\nabla\phi|^2 dV \geq -C + \int \phi^2 \log \phi^2 dV + \frac{d}{2}\log(4\pi\tau) + d.$$

In fact, we can take the

$$\mu(M, g, \tau) = \inf\left\{W(M, g, f, \tau) : \int (4\pi\tau)^{-d/2} e^{-f} dV = 1\right\},$$

which is the best possible constant $-C$. It can be shown that μ is finite, which is what we could call a log-Sobolev inequality.

We can now show that if $g(t)$ is a solution to Ricci flow on $t \in [0, T_0]$ and $\tau = T_0 - t$, then $\mu(M, g, \tau)$ is increasing. The first exercise is important:

Exercise 7 *Show that $\mu(M, g, \tau) = W(M, g, f_*, \tau)$ for a function $f_* \in H^1(M)$. (Not quite true... What is the true statement? Hint: you need to change to a new function ϕ.)*

Once we know that μ is realized by a function, we can show that $\mu(M, g(t), \tau(t))$ is increasing as follows. Calculate $\mu(M, g(t_0), \tau(t_0)) = W(M, g(t_0), f_*(t_0), \tau(t_0))$ for some minimizer $f_*(t_0)$. For any time $t \leq t_0$, we can solve the equation for f in 4 backwards to t with initial condition $f(t_0, x) = f_*(t_0, x)$ (since f_* is in $H^1(M)$, there exists a weak solution to this parabolic flow). We know that

$$\begin{aligned}\mu(M, g(t), \tau(t)) &\leq W(M, g(t), f(t), \tau(t)) \\ &\leq W(M, g(t_0), f_*(t_0), \tau(t_0)) = \mu(M, g(t_0), \tau(t_0)).\end{aligned}$$

5 Noncollapsing

We will now show that log-Sobolev inequalities imply noncollapsing. Suppose we have a ball $B(p, \sqrt{\tau})$ with bounded normalized curvature, i.e.,

$$|\mathrm{Rm}(x)| \leq \frac{1}{\tau}$$

for $x \in B(p, \sqrt{\tau})$. Then $|R|\tau \leq c(d)$ for some constant depending only on dimension. Then the log-Sobolev inequality can be rewritten as

$$c(d)\int \phi^2 dV + \tau \int 4|\nabla\phi|^2 dV \geq \mu(M, g, \tau) + \int \phi^2 \log \phi^2 dV + \frac{d}{2}\log(4\pi\tau) + d.$$

Suppose ϕ is a function supported on $B(p, \sqrt{\tau})$ such that $\int_M \phi^2 dV = 1$. Then Jensen's inequality implies that

$$\begin{aligned}\frac{1}{V(B)}\int_B \phi^2 \log \phi^2 dV &\geq \left(\frac{1}{V(B)}\int_B \phi^2 dV\right)\log\left(\frac{1}{V(B)}\int_B \phi^2 dV\right) \\ &= \frac{1}{V(B)}\log\frac{1}{V(B)},\end{aligned}$$

where $B = B(p, \sqrt{\tau})$. (Recall that Jensen's inequality requires a probability measure.) So

$$\int_M \phi^2 \log \phi^2 dV \geq \log \frac{1}{V(B)}.$$

We now get, for this particular choice of ϕ,

$$4\tau \int |\nabla\phi|^2 dV \geq \mu(M, g, \tau) + \log\frac{\tau^{d/2}}{V(B)} - c'(d).$$

Now we will specialize ϕ even more. Suppose

$$\phi(x) = c\psi\left(\frac{d(x, p)}{\sqrt{\tau}}\right)$$

for some bump function ψ on the real line which is 1 on $[0, 1/2]$ and supported on $[0, 1]$ (technically, we only need half the bump function, which is how I described it). Thus $\phi(x) = c$ on $B(p, \sqrt{\tau}/2)$ and c is such that

$$\int_B \phi^2 dV = 1,$$

so $c \leq V\left(B\left(p, \sqrt{\tau}/2\right)\right)^{-1/2}$. We can choose ϕ so that $|\nabla \phi| \leq c'' c/\sqrt{\tau}$ on the ball (for some constant c''), and so

$$4c'' \frac{V(B)}{V\left(B_{1/2}\right)} \geq \mu(M, g, \tau) + \log \frac{\tau^{d/2}}{V(B)} - c'(d).$$

Finally, we can use a Bishop-Gromov volume comparison theorem:

Theorem 8 (Bishop-Gromov comparison) *If $\left(M^d, g\right)$ is a complete Riemannian manifold with*

$$\mathrm{Rc} \geq (n-1) K g$$

for some $K \in \mathbb{R}$, then for any $p \in M$, the volume ratio

$$\frac{V(B(p, r))}{V_K\left(B\left(p_K, r\right)\right)}$$

is non-increasing as a function of r, where p_K is a point in the d-dimensional simply connected space of constant sectional curvature K, and $V_K\left(B\left(p_K, r\right)\right)$ is the volume of a ball of radius r in that space.

In particular, we have that

$$\frac{V(B)}{V_{-1/\tau}\left(B\left(p_{-1/\tau}, \sqrt{\tau}\right)\right)} \leq \frac{V\left(B_{1/2}\right)}{V_{-1/\tau}\left(B\left(p_{-1/\tau}, \sqrt{\tau}/2\right)\right)},$$

and thus there is a $\alpha = \alpha(\tau, d)$ such that

$$\frac{V(B)}{V\left(B_{1/2}\right)} \leq \alpha.$$

In fact, α is independent of τ since

$$\begin{aligned} V_{-1/\tau}\left(B\left(p_{-1/\tau}, \sqrt{\tau}\right)\right) &= V_{-1}\left(B\left(p_{-1}, 1\right)\right) \\ V_{-1/\tau}\left(B\left(p_{-1/\tau}, \sqrt{\tau}/2\right)\right) &= V_{-1}\left(B\left(p_{-1}, 1/2\right)\right). \end{aligned}$$

Thus there is a constant c''' which depends on d such that

$$c''' - \mu(M, g, \tau) \geq \log \frac{\tau^{d/2}}{V(B)},$$

i.e.,

$$V(B) \geq \left(e^{\mu - c'''}\right) \tau^{d/2},$$

which implies κ-noncollapsing at a scale $\sqrt{\tau}$ for $\kappa = \exp\left(\mu - c'''\right)$. Let's formulate this into a proposition:

Proposition 9 *There is a constant $c = c(d)$ depending only on dimension such that if $\mu\left(M^d, g, \tau\right)$ is finite, then for $\kappa = \exp\left(\mu(M, g, \tau) - c\right)$, the Riemannian manifold (M, g) is κ-noncollapsed at the scale of $\sqrt{\tau}$.*

Let's collect the facts about μ.

Proposition 10 *The following are true about μ:*

1. $\mu(M,g,\tau) > -\infty$ *for any fixed manifold* (M,g) *and* $\tau > 0$.
2. *If* $(M,g(t))$ *satisfies the Ricci flow for* $t \in [0,T_0]$ *and* $\tau(t) = T_0 - t$, *then* $\mu(M,g(t),\tau(t))$ *is increasing.*
3. *There is a constant* $c = c(d)$ *depending only on dimension such that the Riemannian manifold* (M,g) *is* κ*-noncollapsed at the scale of* $\sqrt{\tau}$ *at every point for* $\kappa = \exp(\mu(M,g,\tau) - c)$.

We can now prove:

Theorem 11 (Perelman's noncollapsing theorem, first version) *Let* $(M,g(t))$ *be a solution to the Ricci flow on compact* 3*-manifolds for* $t \in [0,T)$ *such that at* $t = 0$ *we have*

$$|\mathrm{Rm}(p)|_{g(0)} \leq 1$$
$$V\left(B_{g(0)}(p,1)\right) \geq \omega$$

for all $p \in M$ *and* $\omega > 0$ *fixed. For any* $\rho > 0$, *there exists* $\kappa = \kappa(\omega,T,\rho) > 0$ *such that the Ricci flow is* κ*-noncollapsed for all* $(t_0,x_0) \in [0,T) \times M$ *and scales* $0 < r_0 < \rho$. *We could also take* $\rho = \rho(t)$ *and get a similar result, as long as* $\rho(t)$ *is uniformly bounded on* $[0,T)$.

Proof. We already showed that for a given τ and metric, $\mu(M,g,\tau)$ has a lower bound. For any r_0^2, we see by monotonicity that

$$\mu\left(M,g(t),r_0^2\right) \geq \mu\left(M,g(0),r_0^2 + t\right).$$

Thus we have that if

$$\mu_0 = \inf\left\{\mu\left(M,g(0),r^2\right) : r^2 \in (0,\rho+T)\right\}$$

then

$$\mu\left(M,g(t),r_0^2\right) \geq \mu_0.$$

Thus $(M,g(t))$ is κ-noncollapsed at the scale of r_0 for all

$$\kappa = \exp(\mu_0 - c) \leq \exp\left[\mu\left(M,g(t),r_0^2\right) - c\right].$$

We need to see that μ_0 is not $-\infty$. Since T is finite, there is no problem at the top of the interval for r^2. It can be shown that as $r^2 \to 0^+$, $\mu\left(M,g(0),r^2\right) \to 0$ (in the interest of time, we will not show this) and so there is no problem at the other side. ■

Remark 12 *This is a bit stronger than what I proposed in an earlier lecture. I think Tao was thinking about future incarnations of this theorem, which is why he formulated as he did.*