Back to the program

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1 Introduction

I would like to go back to the general program and see what we need to learn about. Much of this if from Morgan-Tian, with help from other sources.

Recall that we wish to perform surgery when the Ricci flow comes to a singularity. We will consider a Ricci flow with surgery $(M, g(t))$ defined for $0 \leq t < T < \infty$ which satisfies the following properties:

1. (Normalized initial conditions) We have
$$|\mathrm{Rm}(g(0))| \leq 1$$
and
$$V(B(x,1)) \geq \frac{1}{2} V(B_{\mathbb{E}^3}(0,1))$$
for any $x \in M$.

2. (Curvature pinching) The curvature is pinched towards positive. This means that as the scalar curvature $R \to +\infty$, the ratio of the absolute value of the smallest eigenvalue of the Riemannian curvature tensor to the largest positive eigenvalue goes to zero.

3. (Noncollapsed) There is a $\kappa > 0$ so that the Ricci flow is κ-noncollapsed.

4. (Canonical neighborhood) Any point with large curvature has a canonical neighborhood.

The key is to show that these conditions both allow surgery and then persist after the surgery. In order to do this, we will need to be more precise with 3 and especially 4.

2 Finding canonical neighborhoods

The main way to find these is to take blow-ups as one goes to a singularity. If one takes a sequence $t_i \to T$, where T is a singular time, then the blow ups

$$g_i(t) = M_i g\left(t_i + \frac{t}{M_i}\right).$$

If M_i is comparable to $\sup |\mathrm{Rm}(g(t_i))|$, then $M_i \to \infty$. Furthermore, since $T < \infty$, we have that $g_i(t)$ is defined on the interval $[-M_i t_i, (T - t_i) M_i)$. Thus the limit will definitely be defined on $(-\infty, 0]$, and is thus ancient. Moreover, if we show that g is κ-noncollapsed at some scale r_0, then $g_i(t)$ will be κ-noncollapsed at a scale $r_0\sqrt{M_i}$, and so the limit is κ-noncollapsed on all scales. Finally, one can show that any 3-dimensional ancient solution has nonnegative curvature.

Definition 1 *A κ-solution of Ricci flow is a solution defined for $t \in (-\infty, 0]$ which is κ-noncollapsed on all scales and has nonnegative curvature.*

These are the limits as you go to a finite time singularity. The key is that Perelman was able to classify these solutions in the following way:

1. κ-solutions look like gradient shrinking solitons as $t \to -\infty$.
2. Gradient shrinking solitons in dimension 3 must have finite covers isometric to (i) 3-spheres or (ii) 2-spheres cross $\mathbb{R}$.
3. κ-solutions have canonical neighborhoods.

Now the idea is that as one goes to a singularity, the manifold is like a κ-solution, and thus has canonical neighborhoods.

3 Canonical neighborhoods

What is a canonical neighborhood? This is where the surgery should be done, so it needs to be classified sufficiently to allow us to do a careful surgery. It turns out that there are 4 types of canonical neighborhoods:

Definition 2 *A point $x \in M$ is in a (C, ε)-canonical neighborhood if one of the following holds:*

1. *x is contained in a C-component.*
2. *x is contained in an open set which is within ε of round in the $C^{\lfloor 1/\varepsilon \rfloor}$-topology.*
3. *x is contained in the core of a (C, ε)-cap.*
4. *x is in the center of a strong neck.*

Definition 3 *Define the $C^k(X, g_0)$ "norm" on (X, g) to be*

$$\|(X,g)\|^2_{C^k(X,g_0)} = \sup_{x\in X}\left\{|g(x) - g_0(x)|^2_{g_0} + \sum_{\ell=1}^{k}\left|\nabla^{\ell}_{g_0} g(x)\right|^2_{g_0}\right\}.$$

Remark 4 *Technically, the norm should be*

$$|||(X,g)|||^2_{C^k(X,g_0)} = \sup_{x\in X}\left\{|g(x)|^2_{g_0} + \sum_{\ell=1}^{k}\left|\nabla^{\ell}_{g_0} g(x)\right|^2_{g_0}\right\}.$$

The function $\|\cdot\|$ defined above is essentially $\|(X,g)\| = |||(X, g - g_0)|||$. With our current definition, $\|(X,g)\|_{C^k(X,g_0)} = 0$ if $g = g_0$.

Problem 5 *Here is something to think about. Fix $X \subset \mathbb{R}^n$, say bounded. Given a sequence of Riemannian metrics $\{g_i\}$ on X, under what conditions does there exist a subsequence such that (M, g_i) converge to some limit (X, g_∞) for some Riemannian metric g_∞ (i.e., $\|(X,g_i)\|_{C^k(X,g_\infty)} \to 0$ as $i \to \infty$.*

Problem 6 *How could one use this norm idea to compare (X, g) and (X', g') where $X \neq X'$?*

Definition 7 *Let (N, g) be a Riemannian manifold and $x \in N$ a point. Then an ε-neck structure on (N, g) centered at x consists of a diffeomorphism*

$$\phi : S^2 \times \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \to N,$$

with $x \in \phi\left(S^2 \times \{0\}\right)$, such that

$$\|(N, R(x)\,\phi^* g)\| < \varepsilon$$

where the norm is with respect to $C^{\lfloor 1/\varepsilon \rfloor}\left(S^2 \times \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right), g_{std}\right)$, where g_{std} is the product of the metric with curvature $1/2$ on S^2 with the Euclidean metric on the interval. We say N is a ε-neck centered at x.

Definition 8 *A compact, connected, Riemannian manifold (M, g) is called a C-component if*

1. *M is diffeomorphic to S^3 or $\mathbb{RP}^3$,*
2. *(M, g) has positive sectional curvature,*
3. *For every 2-plane P in TX,*

$$\frac{1}{C} < \frac{\inf_P K(P)}{\sup_{y\in M} R(y)},$$

4.

$$C^{-1}\sup_{y\in M}\frac{1}{\sqrt{R(y)}}<\operatorname{diam}(M)<C\inf_{y\in M}\frac{1}{\sqrt{R(y)}}.$$

Definition 9 *A compact, connected* 3*-manifold* (M,g) *is within* ε *of round in the* $C^{\lfloor 1/\varepsilon\rfloor}$*-topology if there exists a constant* $\rho>0$*, a compact manifold* (Z,g_0) *of constant curvature* $+1$*, and a diffeomorphism*

$$\phi:Z\to M$$

such that

$$\|(Z,\phi^*(\rho g))\|_{C^{\lfloor 1/\varepsilon\rfloor}(Z,g_0)}\leq\varepsilon.$$

Finally, we have the complicated definition of a cap. The last conditions essentially say that the diameter, volume, and curvature differences are controlled and are technical conditions needed in some arguments.

Definition 10 *Let* (M,g) *be a Riemannian* 3*-manifold. A* (C,ε)*-cap in* (M,g) *is a noncompact submanifold* $(\mathcal{C},g|_{\mathcal{C}})$ *together with an open submanifold* $M\subset\mathcal{C}$ *with the following properties:*

1. $\mathcal{C}$ *is diffeomorphic to an open* 3*-ball or to a punctured* $\mathbb{RP}^3$.
2. N *is a* ε*-neck.*
3. $\bar{Y}=\mathcal{C}\setminus N$ *is a compact submanifold with boundary. Its interior* Y *is called the core of* $\mathcal{C}$.
4. *The scalar curvature* $R(y)>0$ *for every* $y\in\mathcal{C}$ *and*
$$\operatorname{diam}(\mathcal{C},g|_{\mathcal{C}})<\frac{C}{\sqrt{\sup_{y\in\mathcal{C}}R(y)}}.$$
5. $$\sup_{x,y\in\mathcal{C}}\frac{R(x)}{R(y)}<C.$$
6. $$V(\mathcal{C})<\frac{C}{\left(\sup_{y\in\mathcal{C}}R(y)\right)^{3/2}}.$$
7. *For any* $y\in Y$*, let* r_y *defined by the the condition that*
$$\sup_{y'\in B(y,r_y)}R(y')=\frac{1}{r_y^2}.$$
Then for each $y\in Y$ *the ball* $B(y,r_y)$ *has compact closure in* $\mathcal{C}$ *and*
$$\frac{1}{C}<\inf_{y\in Y}\frac{V(B(y,r_y))}{r_y^3}.$$

8.

$$\sup_{y\in\mathcal{C}} \frac{|\nabla R(y)|}{R(y)^{3/2}} < C$$

and

$$\sup_{y\in\mathcal{C}} \frac{\left|\frac{\partial R}{\partial t}(y)\right|}{R(y)^{2}} < C$$

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5 How surgery works

The key observations are this:

1. For every κ and every small $\varepsilon > 0$, there is a $C_1 = C_1(\varepsilon, \kappa)$ such that a κ-solution is the union of (C_1, ε) canonical neighborhoods.
2. For every small $\varepsilon > 0$, there is a $C_2 = C_2(\varepsilon)$ and a standard solution of Ricci flow with is the union of (C_2, ε) canonical neighborhoods.
3. We can do surgery on canonical neighborhoods if they are sufficiently small and positively curved.

Consider a Ricci flow which becomes singular at a time T. Fix $T^- < T$ so that there are no surgeries in the interval $[T^-, T)$. By the assumptions, there is an open set $\Omega \subset M$ such that the curvature is bounded for all $t \in [T^-, T)$, so there is a limiting metric on Ω as $t \to T$. Every end is the end of a canonical neighborhood, which looks like a tube. We call these ends ε-horns. We can then fix a constant ρ and consider the subset $\Omega_\rho \subset \Omega$ in which the scalar curvature is bounded above by ρ^{-2}. One can then show that ε horns with boundaries in Ω_ρ are δ-necks. We then do surgery on these δ-necks by gluing in a standard solution.

6 Some things to prove

Here are some things we will need to do in order for this procedure to work:

1) Derive a description of κ-solutions. This will be with regard to what the asymptotic shrinking soliton is, and so we will need to show that there is one.

2) Show that solutions have canonical neighborhoods.

3) Describe the canonical solution

4) Show finite time extinction.