L-distance

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1 Introduction

We first wish to show that a κ-solution has a limit which is a gradient-shrinking soliton. To do this, we will need to introduce a new notion, the reduced distance.

2 Short discussion of W vs reduced distance

We were able to show quite a bit using the W functional, so why introduce the reduced distance (whatever that is)? Let's first think a bit about the case of Euclidean space. Recall that in Euclidean space, W becomes

$$W(M,g,u,\tau)=\int\left[\tau\frac{|\nabla u|^2}{u^2}-u\log u\right]dx-\frac{d}{2}\log(4\pi\tau)-d$$

and that W is minimized for

$$u_*(x,\tau)=(4\pi\tau)^{-d/2}\exp\left(-|x|^2/(4\tau)\right).$$

We see that this formula essentially gives the distance function $d(x,0)^2=|x|^2$ by

$$\frac{|x|^2}{4\tau}=-\log u_*(x,\tau)-\frac{d}{2}\log(4\pi\tau).$$

However, we have very little control over this function. One can also derive the distance function as follows:

$$d(x,0)=\inf_{\substack{\gamma(0)=0\\ \gamma(a)=x}}\int_0^a|\dot\gamma|\,d\tau.$$

This strong relationship is only true in Euclidean space. In general, there are two different concepts: the heat kernel $u_*(x,\tau)$ and the Riemannian distance function $d(x,x_0)$. The function u_* is essentially defined as the solution to a PDE and the distance function is defined by minimizing over paths. Thus, often $d(x,x_0)$ is easier to work with and easier to get more precise information about. The two things are closely related, but not the same concept. We will try to do something similar for the functional W.

3 $\mathcal{L}$-length and reduced distance

Consider a curve $\gamma : [0, \tau] \to M$. One can define the length of a curve as

$$\ell(\gamma) = \int_0^\tau |\dot{\gamma}(s)|\, ds$$

and the energy as

$$E(\gamma) = \frac{1}{2}\int_0^\tau |\dot{\gamma}(s)|^2\, ds.$$

Note that the length is independent of reparametrization, but the energy is not. These notions give rise to the function r_p which is the function representing the distance to the point p, i.e.,

$$r_p(x) = d(x, p).$$

It turns out that the distance function satisfies some differential equations and inequalities, in particular

$$\triangle r \le \frac{d-1}{r}$$

for Ricci nonnegative. Note that if $r(x) = \sqrt{\sum (x^i)^2}$, then $\triangle r = \frac{d-1}{r}$. We will show this inequality later in this lecture, but for now, let's assume this and see the implications on the volume. Notice that

$$\frac{d}{dR} V(B(p,R)) = A(S(p,R))$$

where $S(p,R)$ is the geodesic sphere of radius R centered at p, and A is the area ($d-1$ dimensional volume measure) function. By Gauss lemma, we have that

$$|\nabla r| = 1$$

and also that we can decompose the metric to be

$$dr^2 + g_{S(p,r)}$$

where $g_{S(p,r)}$ is the metric on the geodesic sphere. Denote the measure on the sphere of radius r as $dA_{S(p,r)}$. We see that

$$\begin{aligned}
\int_{B(p,R)} \triangle r dV &= \int_{S(p,R)} (\nabla r \cdot n)\, dA_{S(p,r)} \\
&= \int_{S(p,R)} |\nabla r|\, dA_{S(p,r)} \\
&= A(S(p,R)).
\end{aligned}$$

The decomposition above implies that the volume measure can be decomposed as

$$dV = dA_{S(p,r)} dr$$

and so

$$\begin{aligned}\frac{d}{dR}\int_{B(p,R)}\triangle r dV &= \frac{d}{dR}\int_0^R \int_{S(p,r)} \triangle r dA_{S(p,r)} dr \\ &= R^{d-1}\int_{S(p,R)} \triangle r dA_{S(p,R)}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{d}{dR}\frac{A\left(S\left(p,R\right)\right)}{R^{d-1}} &= \frac{1}{R^{2(d-1)}}\left[R^{d-1}\frac{d}{dR}\int_{B(p,R)}\triangle r dV-\left(d-1\right)R^{d-2}A\left(S\left(p,R\right)\right)\right] \\ &= \frac{1}{R^{2(d-1)}}\left[R^{d-1}\int_{S(p,R)}\triangle r dA_{S(p,R)}-\left(d-1\right)R^{d-2}A\left(S\left(p,R\right)\right)\right] \\ &\leq \frac{1}{R^{2(d-1)}}\left[\left(d-1\right)R^{d-2}\int_{S(p,R)}dA_{S(p,R)}-\left(d-1\right)R^{d-2}A\left(S\left(p,r\right)\right)\right] \\ &= 0.\end{aligned}$$

This implies an inequality on volume ratios.

Theorem 1 (Bishop-Gromov theorem, simplified version) *If* $\mathrm{Rc}\geq 0$ *then* $V\left(B\left(p,r\right)\right)/r^{d}$ *is a nonincreasing function of* r.

Proof. We have already showed that

$$\frac{d}{dR}\frac{A\left(S\left(p,R\right)\right)}{R^{d-1}}\leq 0.$$

We now consider $0\leq r_1\leq r_2$. We have

$$\begin{aligned}\frac{V\left(B\left(p,r_2\right)\right)-V\left(B\left(p,r_1\right)\right)}{r_2^d-r_1^d} &= \frac{\int_{r_1}^{r_2}A\left(S\left(p,r\right)\right)dr}{\int_{r_1}^{r_2}d\ r^{d-1}dr} \\ &= \frac{\int_{r_1}^{r_2}\frac{A(S(p,r))}{r^{n-1}}r^{d-1}dr}{\int_{r_1}^{r_2}d\ r^{d-1}dr} \\ &\leq \frac{A\left(S\left(p,r_1\right)\right)}{d\ r_1^{d-1}}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{V\left(B\left(p,r_1\right)\right)}{r_1^d} &= \frac{\int_0^{r_1}A\left(S\left(p,r\right)\right)dr}{\int_0^{r_1}d\ r^{d-1}dr} \\ &= \frac{\int_0^{r_1}\frac{A(S(p,r))}{r^{n-1}}r^{d-1}dr}{\int_0^{r_1}d\ r^{d-1}dr} \\ &\geq \frac{A\left(S\left(p,r_1\right)\right)}{d\ r_1^{d-1}}.\end{aligned}$$

Thus we have

$$\frac{V(B(p,r_1))}{r_1^d} \geq \frac{V(B(p,r_2)) - V(B(p,r_1))}{r_2^d - r_1^d},$$

which implies

$$\begin{aligned} 0 &\leq \frac{V(B(p,r_1))}{r_1^d} - \frac{V(B(p,r_2)) - V(B(p,r_1))}{r_2^d - r_1^d} \\ &= \frac{r_2^d V(B(p,r_1)) - r_1^d V(B(p,r_2))}{r_1^d \left(r_2^d - r_1^d\right)} \\ &= \frac{r_2^d}{\left(r_2^d - r_1^d\right)} \left[\frac{V(B(p,r_1))}{r_1^d} - \frac{V(B(p,r_2))}{r_2^d}\right]. \end{aligned}$$

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We now consider Ricci flows. Let $(M, g(\tau))$ be a solution to backwards Ricci flow, i.e.,

$$\frac{\partial g}{\partial \tau} = \mathrm{Rc}(g(\tau))$$

and let $\gamma : [0,\tau] \to M$ be a curve in M. We define the $\mathcal{L}$-distance to be

$$\mathcal{L}(\gamma) = \int_0^\tau \sqrt{\sigma} \left(|\gamma'(\sigma)|_{g(\sigma)}^2 + R_{g(\sigma)} \right) d\sigma.$$

This may need a bit of explanation. First, $\gamma' = \frac{d\gamma}{d\tau}$ is the tangent vector to the curve γ. Note that it is measured at σ by the metric $g(\sigma)$ (so at different ^{s}, it is measured using different metrics!) The first term looks almost like the term in the energy $\int |\gamma'|^2 \, d\sigma$, which can be used to derive geodesics on a Riemannian manifold. However, the addition of the term $\sqrt{\sigma}$ makes the integral scale like length, not energy (think about this).

Remark 2 *Tao uses $-\tau$ in some places because of the understanding that $\tau = -t$ and he wants $g(t)$ to be a solution to Ricci flow and $V(t)$ (defined later) to be defined on t. We will allow g to be parametrized by τ and so there is no need for this.*

In the smooth, fixed manifold case, one considers the length functional

$$L(\gamma) = \int_0^{\tau_1} |\gamma'(\sigma)|_g \, d\sigma$$

and then one can find the distance $d(x_0, x)$ between points by taking the infimum of length over all curves connecting those two points. One can define the distance function $r_{x_0}(x)$ which is the function which returns the distance to a fixed point x_0. The analogue of this in the Ricci flow case is the ℓ-distance (also called reduced length):

$$\ell_{(0,x_0)}(\tau, x) = \frac{1}{2\sqrt{\tau}} \inf \{\mathcal{L}(\gamma)\}$$

where the inf is over all paths γ from x_0 to x.

Remark 3 *Note there is also the $\mathcal{L}$-distance, which is $2\sqrt{\tau}\ell$.*

One can finally define the reduced volume

$$\tilde{V}_{(0,x_0)}(\tau) = \int_M \tau^{-d/2} \exp\left[-\ell_{(0,x_0)}(\tau, x)\right] dV_{g(\tau)}.$$

Our main goal will be to show the following:

Theorem 4

$$\frac{\partial}{\partial \tau}\ell_{(0,x_0)} - \triangle_{g(\tau)}\ell_{(0,x_0)} + \left|\nabla \ell_{(0,x_0)}\right|^2_{g(\tau)} - R + \frac{d}{2\tau} \geq 0.$$

As a corollary, we get monotonicity of the reduced volume if $\frac{\partial}{\partial \tau} g = 2\,\mathrm{Rc}(g)$.

Corollary 5 (Reduced Volume is monotone)

$$\frac{\partial}{\partial \tau}\tilde{V}_{(0,x_0)}(\tau) \leq 0.$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial \tau}\tilde{V}_{(0,x_0)}(\tau) &= \frac{\partial}{\partial \tau}\int_M \tau^{-d/2} \exp\left[-\ell_{(0,x_0)}(\tau, x)\right] dV_{g(\tau)} \\
&= -\frac{d}{2\tau}\tilde{V}_{(0,x_0)}(\tau) - \int_M \frac{\partial \ell_{(0,x_0)}}{\partial \tau}\tau^{-d/2} \exp\left[-\ell_{(0,x_0)}(\tau, x)\right] dV_{g(\tau)} \\
&+ \int_M \tau^{-d/2} \exp\left[-\ell_{(0,x_0)}(\tau, x)\right] R dV_{g(\tau)} \\
&\leq -\frac{d}{2\tau}\tilde{V}_{(0,x_0)}(\tau) + \int_M \tau^{-d/2} \exp\left[-\ell_{(0,x_0)}(\tau, x)\right] R dV_{g(\tau)} \\
&\quad + \int_M \left(-\triangle_{g(\tau)}\ell_{(0,x_0)} + \left|\nabla \ell\right|^2_{g(\tau)} - R + \frac{d}{2\tau}\right)\tau^{-d/2} \exp\left[-\ell_{(0,x_0)}(\tau, x)\right] dV_{g(\tau)} \\
&= 0.
\end{aligned}$$

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Of course, all of this assumes sufficient regularity on ℓ, which is not, in general true. However, this argument can be made rigorous in some generality, including past the "conjugate radius."

4 Variations of length and the distance function

We start with the energy functional given above. We can do calculus of variations as follows. Consider a variation $\Gamma(t, s)$ such that

$$\begin{aligned}
\Gamma(t, s = 0) &= \gamma(t) \\
\frac{\partial}{\partial s}\Gamma(t, s = 0) &= X(t),
\end{aligned}$$

so we consider the variation to be X. Furthermore, we consider variations which fix the endpoints, i.e., $X(0) = X(\tau) = 0$. Compute

$$\begin{aligned}
\delta E_\gamma(X) &= \delta \frac{1}{2} \int_0^\tau g(\dot\gamma, \dot\gamma)\, dt \\
&= \frac{1}{2} \left.\frac{\partial}{\partial s}\right|_{s=0} \int_0^\tau g\left(\frac{\partial \Gamma}{\partial t}, \frac{\partial \Gamma}{\partial t}\right) dt \\
&= \int_0^\tau g\left(D_s \frac{\partial \Gamma}{\partial t}, \frac{\partial \Gamma}{\partial t}\right) dt \\
&= \int_0^\tau g(D_t X, \dot\gamma)\, dt \\
&= g(X, \dot\gamma)|_0^\tau - \int_0^\tau g(X, D_t \dot\gamma)\, dt.
\end{aligned}$$

Remark 6 *The notion $D_t X$ means $D_t X = \nabla_{\frac{\partial}{\partial t}} X$. It only depends on the values along the curve γ. (Why?)*

At a critical point for the energy, we must have $\delta E_\gamma(X) = 0$ for all X, so we get that $D_t \dot\gamma = 0$ if the endpoints are fixed, i.e., if $X(0) = X(\tau) = 0$. This is the geodesic equation. Notice that if we restrict to a geodesic curve, (i.e., $D_t \dot\gamma = 0$) and fix the initial point ($X(0) = 0$), then

$$\delta E_\gamma(X) = g(X(\tau), \dot\gamma(\tau)).$$

On a geodesic,

$$\frac{d}{dt} |\dot\gamma(t)|^2 = 2g(D_t \dot\gamma, \dot\gamma) = 0$$

so γ has constant velocity. So along a geodesic, we have

$$\begin{aligned}
E(\gamma) &= \frac{1}{2} \int_0^\tau |\dot\gamma|^2\, ds = \frac{1}{2} |\dot\gamma|^2 \tau \\
\ell(\gamma) &= \int_0^\tau |\dot\gamma|\, ds = |\dot\gamma| \tau
\end{aligned}$$

and so if γ is a geodesic, then

$$E(\gamma) = \frac{1}{2\tau} [\ell(\gamma)]^2$$

and so we have that if γ is the gives the distance $d(p, x)$, then

$$E(\gamma) = \frac{1}{2\tau} d(p, x)^2.$$

If we want to compute a variation

$$\frac{d}{ds} d(p, x(s))^2$$

away from the cut locus (so variations in geodesics stay minimizing), we have

$$\frac{d}{ds} d\left(p, x\left(s\right)\right)^2 = 2\tau \delta E_\gamma\left(X\right) = 2\tau g\left(\frac{dx}{ds}, \dot\gamma\left(\tau\right)\right),$$

where X is a variation of geodesics (one must show that these exist, but they do!) Note that we can always reparametrize so that $\tau = 1$, and then we get that

$$\left|\nabla d\left(p, x\right)^2\right|^2 = 4\left|\dot\gamma\left(1\right)\right|^2,$$

or

$$4d\left(p, x\right)^2 \left|\nabla d\left(p, x\right)\right|^2 = 4\left|\dot\gamma\left(1\right)\right|^2 = 4d\left(p, x\right)^2$$

and so

$$\left|\nabla d\left(p, x\right)\right| = 1.$$

Now compute the second variation of energy when γ is a geodesic. We get

$$\begin{aligned}\delta^2 E_\gamma\left(X, X\right) &= \frac{\partial}{\partial s}\int_0^\tau g\left(D_t X, \dot\gamma\right) dt\\ &= g\left(X, \dot\gamma\right)\big|_0^\tau - \int_0^\tau g\left(X, D_t\dot\gamma\right) dt.\end{aligned}$$

$$\begin{aligned}\delta^2 E_\gamma\left(X, X\right) &= \frac{1}{2}\left.\frac{\partial^2}{\partial s^2}\right|_{s=0}\int_0^\tau g\left(\frac{\partial\Gamma}{\partial t}, \frac{\partial\Gamma}{\partial t}\right) dt\\ &= \int_0^\tau g\left(D_s D_s \frac{\partial\Gamma}{\partial t}, \frac{\partial\Gamma}{\partial t}\right) + g\left(D_s\frac{\partial\Gamma}{\partial t}, D_s\frac{\partial\Gamma}{\partial t}\right) dt\\ &= \int_0^\tau g\left(D_s D_t \frac{\partial\Gamma}{\partial s}, \frac{\partial\Gamma}{\partial t}\right) + g\left(D_t\frac{\partial\Gamma}{\partial s}, D_t\frac{\partial\Gamma}{\partial s}\right) dt\\ &= \int_0^\tau g\left(D_t D_s \frac{\partial\Gamma}{\partial s}, \frac{\partial\Gamma}{\partial t}\right) + g\left(R\left(\frac{\partial\Gamma}{\partial s}, \frac{\partial\Gamma}{\partial t}\right)\frac{\partial\Gamma}{\partial s}, \frac{\partial\Gamma}{\partial t}\right) + g\left(D_t X, D_t X\right) dt.\\ &= g\left(\nabla_X X, \gamma\right)\big|_0^\tau + \int_0^\tau g\left(R\left(X, \dot\gamma\right) X, \dot\gamma\right) + g\left(D_t X, D_t X\right) dt\end{aligned}$$

The first term is zero if the variation is a variation of geodesics at the endpoints (or fixed endpoints). Note our only assumptions are this and that γ is a geodesic.

Let's again parametrize our geodesic between 0 and 1 and fix $\gamma\left(0\right)$ and vary the other endpoint $x\left(s\right)$ so that $X\left(t\right) = \frac{\partial\Gamma}{\partial s}\left(t, 0\right)$ "varies through geodesics." Then

$$\frac{1}{2}\frac{d^2}{ds^2} d\left(p, x\left(s\right)\right)^2 = \int_0^1 g\left(R\left(X, \dot\gamma\right) X, \dot\gamma\right) + g\left(D_t X, D_t X\right) dt.$$

Furthermore, we can take $X\left(t\right) = t\frac{\partial x}{\partial s}$, where $\frac{\partial x}{\partial s}$ is the parallel translation along γ of $\frac{\partial x}{\partial s}$. Then we get

$$\frac{1}{2}\frac{d^2}{ds^2} d\left(p, x\left(s\right)\right)^2 = \int_0^1 t^2 g\left(R\left(\frac{\partial x}{\partial s}, \dot\gamma\right)\frac{\partial x}{\partial s}, \dot\gamma\right) dt + g\left(\frac{\partial x}{\partial s}, \frac{\partial x}{\partial s}\right).$$

Thus if we sum over an orthonormal frame and assume that the sectional curvature is nonnegative, we have that

$$\frac{1}{2}\triangle d(p,x)^2 \leq d.$$

And so

$$\frac{1}{2}\triangle\left(f^2\right) = f\triangle f + |\nabla f|^2$$

and so

$$\triangle d(p,x) \leq \frac{d-1}{d(p,x)}$$

5 Variations of the reduced distance

We will do the same thing to the reduced distance. Consider

$$\mathcal{L}(\gamma) = \int_0^\tau \sqrt{\sigma}\left(\left|\gamma'(\sigma)\right|^2_{g(\sigma)} + R_{g(\sigma)}\right) d\sigma.$$

Actually, we want to think of this as a functional on spacetime paths. A spacetime path γ is a map

$$\gamma : [0,\tau] \to M \times I$$

where I is an interval. We will only consider paths that look like

$$\gamma(s) = (\tilde{\gamma}(s), s). \tag{1}$$

We can naturally define $R(\gamma(s)) = R_{g(s)}(\tilde{\gamma}(s))$. We then have the formulation of the functional as

$$\mathcal{L}(\gamma) = \int_0^\tau \sqrt{\sigma}\left(\left|\tilde{\gamma}'(\sigma)\right|^2_{g(\sigma)} + R(\gamma(\sigma))\right) d\sigma.$$

We will define the reduced length as

$$\ell_{(0,x_0)}(\tau,x) = \frac{1}{2\sqrt{\tau}} \inf\left\{\mathcal{L}(\gamma) : \gamma \text{ are of the form (1)}\right\}.$$

Note that the infimum can also be considered as the infimum over all paths $\tilde{\gamma}$ on M.

Note that the tangent space of $M \times I$ splits as $TM \times TI$ and TI is spanned by $\frac{\partial}{\partial\tau}$. There is a notion of a horizontal vector field, which is a vector field X such that $d\tau(X) = 0$. Since variations of paths of the form (1) must be horizontal vector fields, we first consider the first variation of $\mathcal{L}$ with respect to a horizontal vector field X. Note that $\tilde{\gamma}' = \gamma' - \frac{\partial}{\partial\tau}$ is a horizontal vector field (but γ' is not).

Remark 7 *Horizontal vector fields on $M \times I$ are in one-to-one correspondence with vector fields on M. For this reason, we may abuse notation and use the same notation for both vector fields. In particular, $\tilde{\gamma}'$ will be considered both a vector field on M and a horizontal vector field on $M \times I$.*

We get

$$\delta\mathcal{L}_\gamma(X)=\int_0^\tau\sqrt{\sigma}\left(2\left\langle\nabla_X\tilde{\gamma}',\tilde{\gamma}'\right\rangle_{g(\sigma)}+\nabla_X R_{g(\sigma)}\right)d\sigma.$$

We do the same thing we did with the energy functional, finding

$$\begin{aligned}\delta\mathcal{L}_\gamma(X)&=\int_0^\tau\sqrt{\sigma}\left(2\left\langle\nabla_{\tilde{\gamma}'}X,\tilde{\gamma}'\right\rangle_{g(\sigma)}+\nabla_X R_{g(\sigma)}\right)d\sigma\\&=\int_0^\tau\sqrt{\sigma}\left(2\frac{d}{d\sigma}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)}-4\operatorname{Rc}\left(X,\tilde{\gamma}'\right)-2\left\langle X,\nabla_{\tilde{\gamma}'}\tilde{\gamma}'\right\rangle_{g(\sigma)}+\nabla_X R_{g(\sigma)}\right)d\sigma\end{aligned}$$

since if X and Y are horizontal,

$$\frac{d}{d\tau}g(X,Y)=2\operatorname{Rc}(X,Y)+g\left(D_{\tilde{\gamma}'}X,Y\right)+g\left(X,D_{\tilde{\gamma}'}Y\right).$$

Remark 8 *In general, the formula is*

$$\frac{d}{d\tau}g(X,Y)=\frac{\partial}{\partial\tau}g(X,Y)+\nabla_{\tilde{\gamma}'}g(X,Y),$$

which has more terms since $\nabla_{\tilde{\gamma}'}X\neq 0$, *etc.*

Remark 9 *I used the notation of total derivative since there is the variation of the metric with respect to the time parameter of Ricci flow and also the variation with respect to* γ'. *I have also used* $'$ *instead of dot to denote derivative with respect to* τ *or* σ.

Now we need to take the $\frac{d}{d\sigma}$ out, so we need that

$$\frac{d}{d\sigma}\left(2\sqrt{\sigma}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)}\right)=\frac{1}{\sqrt{\sigma}}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)}+2\sqrt{\sigma}\frac{d}{d\sigma}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)},$$

so

$$\begin{aligned}\delta\mathcal{L}_\gamma(X)&=\int_0^\tau\frac{d}{d\sigma}\left(2\sqrt{\sigma}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)}\right)-\frac{1}{\sqrt{\sigma}}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)}\\&\quad+\sqrt{\sigma}\left(-4\operatorname{Rc}\left(X,\tilde{\gamma}'\right)-2\left\langle X,\nabla_{\tilde{\gamma}'}\tilde{\gamma}'\right\rangle_{g(\sigma)}+\nabla_X R_{g(\sigma)}\right)d\sigma\\&=2\sqrt{\sigma}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)}\Big|_0^\tau-2\sqrt{\sigma}\int_0^\tau\left\langle G,X\right\rangle_{g(\sigma)}d\sigma\end{aligned}$$

where G is the vector field

$$G(\sigma)=\nabla_{\tilde{\gamma}'}\tilde{\gamma}'+\frac{1}{2\sigma}\tilde{\gamma}'+2\operatorname{Rc}\left(\tilde{\gamma}'\right)-\frac{1}{2}\nabla R_{g(\sigma)}$$

where $\operatorname{Rc}(X)$ is the vector field on M such that

$$\left\langle\operatorname{Rc}(X),Y\right\rangle=\operatorname{Rc}(X,Y)$$

for all for all vector fields Y on M. Notice that G does not depend on the variation X. Note that if γ is a minimizer with fixed endpoints (i.e., for all variations X such that $X(0)=X(\tau)=0$), then we must have that $G=0$. This is the $\mathcal{L}$-geodesic equation.

Problem 10 *If we are in Euclidean space, what are the $\mathcal{L}$-geodesics?*

Now supposing there is a unique minimizer and that ℓ is a smooth function, we can consider variations through $\mathcal{L}$-geodesics such that $X(0)$ is fixed to get

$$\delta\mathcal{L}_\gamma(X) = 2\sqrt{\tau}\,\langle X, \tilde{\gamma}'\rangle_{g(\tau)},$$

which implies that

$$\frac{\partial}{\partial s}\ell_{(0,x_0)}(\tau, x(s)) = \left\langle \frac{\partial x}{\partial s}, \tilde{\gamma}' \right\rangle_{g(\tau)}$$

(recall that ℓ divides by $\sqrt{\tau}$). This can also be written as

$$\nabla \ell_{(0,x_0)} = \tilde{\gamma}'.$$

Note that the reduced length also depends on time, so let's compute the time derivative as well:

$$\delta\mathcal{L}_\gamma\left(\frac{d}{d\tau}\right) = \sqrt{\tau}\left(|\tilde{\gamma}'(\tau)|^2_{g(\tau)} + R_{g(\tau)}\right).$$

Note that this need not be zero since we are not minimizing in this direction. We find that

$$\frac{d}{d\tau}\left(2\sqrt{\tau}\ell_{(0,x_0)}\right) = \sqrt{\tau}\left(|\tilde{\gamma}'(\tau)|^2_{g(\tau)} + R_{g(\tau)}\right).$$

Now, the total derivative decomposes as

$$\frac{d}{d\tau}\ell_{(0,x_0)} = \frac{\partial}{\partial\tau}\ell_{(0,x_0)} + \nabla_{\tilde{\gamma}'}\ell_{(0,x_0)},$$

and so

$$\frac{\partial}{\partial\tau}\ell_{(0,x_0)} = \frac{1}{2}\left(|\tilde{\gamma}'(\tau)|^2_{g(\tau)} + R_{g(\tau)}\right) - \frac{1}{2\tau}\ell_{(0,x_0)} - \frac{1}{2\sqrt{\tau}}\,|\tilde{\gamma}'|^2_{g(\tau)}.$$