Derivatives of distance function

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Math 538, Spring 2009

April 14, 2009

We now know that for a curve $x(s)$,

$$\frac{1}{2}d(x_0, x(s))^2 = E(\gamma_s)$$

where $\gamma_s(t)$ are curves such that

$$D_t\gamma_s' = \nabla_{\gamma_s'}\gamma_s' = 0$$

for all s. We may parametrize all curves between 0 and 1. Furthermore, we know that if we consider the variation $\Gamma(s,t) = \gamma_s(t)$, then

$$\begin{aligned}\left.\frac{\partial}{\partial s}\right|_{s=0}\gamma_s(t) &= X(t), \\ X(0) &= 0, \\ X(1) &= \frac{dx}{ds}.\end{aligned}$$

$$E(\gamma_s) = E(\gamma_0) + s\delta E_{\gamma_0}(X) + \frac{1}{2}s^2\delta^2 E_{\gamma_0}(X,X) + O\left(s^3\right),$$

where

$$\delta^2 E_{\gamma_0}(X,X) = \frac{d^2}{ds^2}E(\gamma_s).$$

Thus the first derivative is

$$\begin{aligned}\left.\frac{d}{ds}\right|_{s=0}\frac{1}{2}d(x_0, x(s))^2 &= \lim_{s\to 0}\frac{\frac{1}{2}d(x_0,x(s))^2 - \frac{1}{2}d(x_0,x(0))^2}{s} \\ &= \delta E_{\gamma_0}(X).\end{aligned}$$

The key fact is that under such a variation, we have

$$\delta E_{\gamma_0}(X) = g(\gamma_0'(1), X(1)) = g\left(\gamma_0'(1), \frac{dx}{ds}(0)\right)$$

so we did not need to know much about $X(s)$ in general!

Now, to compute the second derivative, we look at

$$\left.\frac{d^2}{ds^2}\right|_{s=0} \frac{1}{2} d\left(x_0, x(s)\right)^2 = \lim_{s\to 0} \frac{\frac{1}{2} d\left(x_0, x(s)\right)^2 + \frac{1}{2} d\left(x_0, x(-s)\right)^2 - 2\frac{1}{2} d\left(x_0, x(0)\right)^2}{s^2}$$
$$= \delta^2 E_{\gamma_0}(X, X).$$

$$\begin{aligned}
\delta^2 E_\gamma (X, X) &= \frac{1}{2} \left.\frac{\partial^2}{\partial s^2}\right|_{s=0} \int_0^1 g\left(\frac{\partial \Gamma}{\partial t}, \frac{\partial \Gamma}{\partial t}\right) dt \\
&= \int_0^1 g\left(D_s D_s \frac{\partial \Gamma}{\partial t}, \frac{\partial \Gamma}{\partial t}\right) + g\left(D_s \frac{\partial \Gamma}{\partial t}, D_s \frac{\partial \Gamma}{\partial t}\right) dt \\
&= \int_0^1 g\left(D_s D_t \frac{\partial \Gamma}{\partial s}, \frac{\partial \Gamma}{\partial t}\right) + g\left(D_t \frac{\partial \Gamma}{\partial s}, D_t \frac{\partial \Gamma}{\partial s}\right) dt \\
&= \int_0^1 g\left(D_t D_s \frac{\partial \Gamma}{\partial s}, \frac{\partial \Gamma}{\partial t}\right) + g\left(R\left(\frac{\partial \Gamma}{\partial s}, \frac{\partial \Gamma}{\partial t}\right)\frac{\partial \Gamma}{\partial s}, \frac{\partial \Gamma}{\partial t}\right) + g\left(D_t X, D_t X\right) dt. \\
&= \left. g\left(D_s \frac{\partial \Gamma}{\partial s}, \frac{\partial \Gamma}{\partial t}\right)\right|_0^1 + \int_0^1 g\left(R\left(X, \gamma'\right) X, \gamma'\right) + g\left(D_t X, D_t X\right) dt \\
&= -\int_0^1 K\left(X, \gamma'\right) |X|^2 |\gamma'|^2 + \int_0^1 g\left(D_t X, D_t X\right) dt
\end{aligned}$$

Since I can also choose the variation so that $D_s \frac{\partial \Gamma}{\partial s}(s, t = 1) = 0$.

Fact: The vector field $X = tv$ where v is the parallel transport of $X(1) = \frac{\partial x}{\partial s}$ along the curve γ. This implies that

$$\begin{aligned}
\delta^2 E_\gamma (X, X) &= -\int_0^1 K\left(X, \gamma'\right) |X|^2 |\gamma'|^2 + \int_0^1 g\left(D_t X, D_t X\right) dt \\
&= -\int_0^1 K\left(X, \gamma'\right) |X|^2 |\gamma'|^2 + \left|\frac{\partial x}{\partial s}\right|^2.
\end{aligned}$$

Now if we take $\left| \left.\frac{\partial x}{\partial s}\right|_{s=0} \right| = 1$, then we have

$$\left.\frac{d^2}{ds^2}\right|_{s=0} \frac{1}{2} d\left(x_0, x(s)\right)^2 \leq \left|\frac{\partial x}{\partial s}\right|^2 = 1$$

if the sectional curvatures are positive. Now take normal coordinates at x. At the center of normal coordinates,

$$\triangle f(0) = \left(\frac{\partial}{\partial x^1}\right)^2 f + \cdots \left(\frac{\partial}{\partial x^d}\right)^2 f.$$

So, taking $x(s) = \exp_x\left(s \frac{\partial}{\partial x^i}\right)$, we conclude that

$$\triangle \left[\frac{1}{2} d\left(x_0, x\right)^2\right] \leq d.$$

Furthermore, we have

$$\begin{aligned}\triangle\left[\frac{1}{2}d\left(x_0,x\right)^2\right] &= \nabla\cdot\left(d\left(x_0,x\right)\nabla d\left(x_0,x\right)\right)\\ &= d\left(x_0,x\right)\triangle d\left(x_0,x\right)+\left|\nabla d\left(x_0,x\right)\right|^2\\ &= d\left(x_0,x\right)\triangle d\left(x_0,x\right)+1.\end{aligned}$$

Thus, we get

$$\begin{aligned}d\left(x_0,x\right)\triangle d\left(x_0,x\right)+1 &\leq d\\ \triangle d\left(x_0,x\right) &\leq \frac{d-1}{d\left(x_0,x\right)}.\end{aligned}$$