Notes from Math 538: Ricci flow and the Poincare conjecture

David Glickenstein

Spring 2009

Contents

Preface

These are notes from a topics course on Ricci flow and the Poincaré Conjecture from Spring 2008.

Chapter 1

Introduction

These are notes from a topics course on Ricci flow and the Poincaré Conjecture from Spring 2008.

Chapter 2

Three-manifolds and the Poincaré conjecture

2.1 Introduction

This lecture is mostly taken from Tao's lecture 2.

In this lecture we are going to introduce the Poincaré conjecture and the idea of the proof. The rest of the class will be going into different details of this proof in varying amounts of careful detail. All manifolds will be assumed to be without boundary unless otherwise specified.

The Poincaré conjecture is this:

Theorem 1 (Poincare conjecture) *Let M be a compact 3-manifold which is connected and simply connected. Then M is homeomorphic to the 3-sphere $\mathbb{S}^3$.*

Remark 2 *A simply connected manifold is necessarily orientable.*

In fact, one can prove a stronger statement called Thurston's geometrization conjecture, which is quite a bit more complicated, but is roughly the following:

Theorem 3 (Thurston's geometrization conjecture) *Every 3-manifold M can be cut along spheres and π_1-essential tori such that each piece can be given one of 8 geometries ($\mathbb{E}^3, \mathbb{S}^3, \mathbb{H}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \widetilde{\mathrm{SL}}(2, \mathbb{R}), \mathrm{Nil}, \mathrm{Sol}$).*

We may go into this conjecture a little more if we have time, but certainly elements of this will come up in the process of these lectures. We will take a quick look

2.2 Examples of 3-manifolds

Here we look in the topological category. It turns out that in three dimensions, the smooth category, the piecewise linear category, and the topological category are the same (e.g., any homeomorphism can be approximated by a diffeomorphism, any topological manifold can be given a smooth structure, etc.)

2.2.1 Spherical manifolds

The most basic 3-manifold is the 3-sphere, S^3, which can be constructed in several ways, such as:

- The set of unit vectors in $\mathbb{R}^4$
- The one point compactification of $\mathbb{R}^3$.

Using the second definition, it is clear that any loop can be contracted to a point, and so S^3 is simply connected. One can also look at the first definition and see that the rotations $SO(4)$ of $\mathbb{R}^4$ act transitively on S^3, with stabilizer $SO(3)$, and so S^3 is a homogeneous space described by the quotient $SO(4)/SO(3)$. One can also notice that the unit vectors in $\mathbb{R}^4$ can be given the group structure of the unit quaternions, and it is not too hard to see that this group is isomorphic to $\mathrm{SU}(2)$, which is the double cover of $SO(3)$.

As more examples of 3-manifolds, it is possible to find finite groups Γ acting freely on S^3, and consider quotients of S^3 by the action. Note that if the action is nontrivial, then these new spaces are not simply connected. One can see this in several ways. The direct method is that since there must be $g \in \Gamma$ and $x \in S^3$ such that $gx \neq x$. A path γ from x to gx in S^3 descends to a loop in S^3/Γ. If there is a homotopy of that loop to a point, then one can lift the homotopy to a homotopy $H : [0,1] \times [0,1] \to S^3$ such that $H(t,0) = \gamma(t)$. But then, looking at $H(1,s)$, we see that $H(1,0) = gx$ and $H(1,1) = x$ and $H(1,s) = g'x$ for some $g' \in \Gamma$ for all s. But since Γ is discrete, this is impossible. A more high level approach shows that the map $S^3 \to S^3/\Gamma$ is a covering map, and, in fact, the universal covering map and so $\pi_1\left(S^3/\Gamma\right) = \Gamma$ if Γ acts effectively. Hence each of these spaces S^3/Γ are different manifolds than S^3. They are called spherical manifolds.

The elliptization conjecture states that spherical manifolds are the only manifolds with finite fundamental group.

2.2.2 Sphere bundles over S^1

The next example is to consider S^2 bundles over S^1, which is the same as $S^2 \times [0,1]$ with $S^2 \times \{0\}$ identified with $S^2 \times \{1\}$ by a homeomorphism. Recall that a homeomorphism

$$\phi : S^2 \to S^2$$

induces an isomorphism on homology (or cohomology),

$$\phi_* : H_2\left(S^2\right) \to H_2\left(S^2\right)$$

and since

$$H_2\left(S^2\right) \cong \mathbb{Z}$$

there are only two possibilities for ϕ_*. In fact, these classify the possible maps up to continuous deformation, and so there is an orientation preserving and an orientation reversing homeomorphism and that is all (using π_2 instead of H_2). (See, for instance, Bredon, Cor. 16.4 in Chapter 2.)

Notice that these manifolds have a map $\phi : M \rightarrow S^1$. It is clear that this induces a surjective homomorphism on fundamental group $\phi_* : \pi_1(M) \rightarrow \pi_1(S^1) \cong \mathbb{Z}$. The kernel of this map consists of loops which can be deformed to maps only on S^2 (constant on the other component), and since S^2 is simply connected, this map is an isomorphism. Note that these manifolds are thus not simply-connected.

2.2.3 Connected sum

One can also form new manifolds via the connected sum operation. Given two 3-manifolds, M and M', one forms the connected sum by removing a disk from each manifold and then identifying the boundary of the removed disks. We denote this as $M\#M'$. Recall that in 2D, all manifolds can be formed from the sphere and the torus in this way.

Now, we may consider the class of all compact, connected 3-manifolds (up to homeomorphism) with the connected sum operation. These form a monoid (essentially a group without inverse), with an identity (S^3). Any nontrivial (i.e., non-identity) manifold can be decomposed into pieces by connected sums, i.e., given any M, we can write

$$M \approx M_1 \# M_2 \# \cdots \# M_k$$

where M_j cannot be written as a connected sum any more (this is a theorem of Kneser). We call such a decomposition a prime decomposition and such manifolds M_j prime manifolds. The proof is very similar to the fundamental theorem of arithmetic which gives the prime decomposition of positive integers.

Proposition 4 *Suppose M and M' are connected manifolds of the same dimension. Then*

1. *$M\#M'$ is compact if and only if both M and M' are compact.*
2. *$M\#M'$ is orientable if and only if both M and M' are orientable.*
3. *$M\#M'$ is simply connected if and only if both M and M' are simply connected.*

We leave the proof as an exercise, but it is not too difficult.

2.3 Idea of the Hamilton-Perelman proof

In order to give the idea, we will introduce a few concepts which will be defined more precisely in successive lectures.

Any smooth manifold can be given a Riemannian metric, denoted g_{ij} or $g(\cdot,\cdot)$, which is essentially an inner product (i.e., symmetric, positive-definite bilinear form) at each tangent space which varies smoothly as the basepoint of the tangent space changes. The Riemannian metric allows one to define angles between two curves and also to measure lengths of (piecewise C^1-) curves by integrating the tangent vectors of a curve. That is, if $\gamma : [0,a] \to M$ is a curve, we can calculate its length as

$$\ell(\gamma) = \int_0^a g(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2}\, dt,$$

where $\dot{\gamma}(t)$ is the tangent to the curve at t. A Riemannian metric induces a metric space structure on M, as the distance between two points is given by the infimum of lengths of all piecewise smooth curves from one point to the other. It is a fact that the metric topology induces the original topology of the manifold.

The main idea is to deform any Riemannian metric to a standard one. This is the idea of "geometrizing." How does one choose the deformation? R. Hamilton first proposed to deform by an equation called the Ricci flow, which is a partial differential equation defined by

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} = -2\,\mathrm{Rc}(g_{ij})$$

where t is an extra parameter (not related to the original coordinates, so $g_{ij} = g_{ij}(t,x)$, where x are the coordinates) and $R_{ij} = \mathrm{Rc}(g_{ij})$ is the Ricci curvature, a differential operator (2nd order) on the Riemannian metric. That means that the Ricci flow equation is a partial differential equation on the Riemannian metric. In coordinates, it roughly looks like

$$\frac{\partial}{\partial t} g_{ij} = -2g^{k\ell}\frac{\partial^2}{\partial x^k \partial x^\ell} g_{ij} + F(g_{ij}, \partial g_{ij})$$

where $g^{k\ell}$ is the inverse matrix of g_{ij} and F is a function depending only on the metric and first derivatives of the metric.

2.3.1 Hamilton's first result

The idea is that as the metric evolves, its curvature becomes more and more uniform. It was shown in Hamilton's landmark 1982 paper that

Theorem 5 (Hamilton) *Given a Riemannian 3-manifold (M, g_0) with positive Ricci curvature, then the Ricci flow with $g(0) = g_0$ exists on a maximal time interval $[0, t_*)$. Furthermore, the Ricci curvature of the metrics $g(t)$ become increasingly uniform as $t \to t_*$. More precisely,*

$$R_{ij}(t) - \bar{R}(t)\, g_{ij}(t),$$

where $\bar{R}$ is the average scalar curvature, converges uniformly to zero as $t \to t_0$.

From this, one can easily show that a rescaling of the metric converges to the round sphere.

2.3.2 2D case

In the 2D case it can be shown that any compact, orientable Riemannian manifold converges under a renormalized Ricci flow (renormalized by rescaling the metric and rescaling time) to a constant curvature metric. This is primarily due to Hamilton, with one case finished by B. Chow. It is possible to use this method to prove the uniformization theorem, which states that any compact, orientable Riemannian manifold can be conformally deformed to a metric with constant curvature. (The original proofs of Hamilton and Chow use the uniformization theorem, but a recent article by Chen, Lu, and Tian shows how to avoid that).

2.3.3 General case

Hamilton introduced a program to study all 3-manifolds using the Ricci flow. It was discovered quite early that the Ricci flow may develop singularities even in the case of a sphere if the Ricci curvature is not positive. An example is the so-called neck pinch singularity. Hamilton's idea was to do surgery at these singularities, then continue the flow and continue to do this until no more surgeries are necessary. Perelman's work describes what happens to the Ricci flow near a singularity and also how to perform the surgery. The new flow is called *Ricci Flow with surgery.* The main result of Perelman is the following.

Theorem 6 (Existence of Ricci flow with surgery) *Let (M,g) be a compact, orientable Riemannian 3-manifold. Then there exists a Ricci flow with surgery $t \rightarrow (M(t), g(t))$ for all $t \in [0,\infty)$ and a closed set $T \subset [0,\infty)$ of surgery times such that:*

1. *(Initial data) $M(0) = M$, $g(0) = g$.*

2. *(Ricci flow) If I is any connected component of $[0,\infty) \setminus T$ (and thus an interval), then $t \rightarrow (M(t), g(t))$ is the Ricci flow on I (you can close this interval on the left endpoint if you wish).*

3. *(Topological compatibility) If $t \in T$ and $\varepsilon > 0$ is sufficiently small, then we know the topological relationship $M(t-\varepsilon)$ and $M(t)$.*

4. *(Geometric compatibility) For each $t \in T$, the metric $g(t)$ on $M(t)$ is related to a certain limit of the metric $g(t-\varepsilon)$ on $M(t-\varepsilon)$ by a certain surgery procedure.*

Note, we can express the topological compatibility more precisely. We have that $M(t-\varepsilon)$ is homeomorphic to the connected sum of finitely many connected components of $M(t)$ together with a finite number of spherical space forms (spherical manifolds), $\mathbb{RP}^3 \# \mathbb{RP}^3$, and $S^2 \times S^1$. Furthermore, each connected component of $M(t)$ is used in the connected sum decomposition of exactly one component of $M(t-\varepsilon)$.

Remark 7 *The case of* $\mathbb{RP}^3 \# \mathbb{RP}^3$ *is interesting in that it is apparently the only nonprime 3-manifold which admits a geometric structure (i.e., is covered by a model geometry; it is a quotient of* $S^2 \times \mathbb{R}$*; this does not contradict our argument above because it is not a sphere bundle over* S^1*). I have seen this mentioned several places, but I do not have a reference.*

Remark 8 *Morgan-Tian and Tao give a more general situation where nonorientable manifolds are allowed. This adds some extra technicalities which we will avoid in this class.*

The existence needs something more to show the Poincaré conjecture. One needs that the surgeries are only discrete and that the flow shrinks everything in finite time.

Theorem 9 (Discrete surgery times) *Let* $t \to (M(t), g(t))$ *be a Ricci flow with surgery starting with an orientable manifold* $M(0)$*. Then the set* T *of surgery times is discrete. In particular, any compact time interval contains a finite number of surgeries.*

Theorem 10 (Finite time extinction) *Let* (M, g) *be a compact* 3*-manifold which is simply connected and let* $t \to (M(t), g(t))$ *be an associated Ricci flow with surgery. Then* $M(t)$ *is empty for sufficiently large* t*.*

With these theorems, one can conclude the Poincaré conjecture in the following way. Given M a simply connected, connected, compact Riemannian manifold, associate a Ricci flow with surgery. It has finite extinction time, and hence finite surgery times. Now one can use the topological decomposition to work backwards and build the manifold backwards, which says that the manifold M is the connected sum of finitely many spherical space forms, copies of $\mathbb{RP}^3 \# \mathbb{RP}^3$, and $S^2 \times S^1$. But since M is the simply connected, everything in the connected sum must be simply connected, and hence every piece of the connected sum must be simply connected, so M must be a sphere.

Chapter 3

Background in differential geometry

3.1 Introduction

We will try to get as quickly as possible to a point where we can do some geometric analysis on Riemannian spaces. One should look at Tao's lecture 0, though I will not follow it too closely.

3.2 Basics of tangent bundles and tensor bundles

Recall that for a smooth manifold M, the tangent bundle can be defined in essentially 3 different ways ((U_i, ϕ_i) are coordinates)

- $TM = \bigsqcup_i (U_i \times \mathbb{R}^n) / \sim$ where for $(x, v) \in U_i \times \mathbb{R}^n$, $(y, w) \in U_j \times \mathbb{R}^n$ we have $(x, v) \sim (y, w)$ if and only iff $y = \phi_j \phi_i^{-1}(x)$ and $w = d\left(\phi_j \phi_i^{-1}\right)_x (v)$.
- $T_pM = \{\text{paths } \gamma : (-\varepsilon, \varepsilon) \to M \text{ such that } \gamma(0) = p\} / \sim$ where $\alpha \sim \beta$ if $(\phi_i \circ \alpha)'(0) = (\phi_i \circ \beta)'(0)$ for every i such that $p \in U_i$. $TM = \bigsqcup_{p \in M} T_pM$.
- T_pM to be the set of derivations of germs at p, i.e., the set of linear functionals X on the germs at p such that $X(fg) = X(f)\,g(p) + f(p)\,X(g)$ for germs f, g at p. $TM = \bigsqcup_{p \in M} T_pM$.

On can define the cotangent bundle by essentially taking the dual of T_pM at each point, which we call T_p^*M, and taking the disjoint union of these to get the cotangent bundle T^*M. One could also use an analogue of the first definition, where the only difference is that instead of using the vector space $\mathbb{R}^n$, one uses its dual and the equivalence takes into account that the dual space pulls back rather than pushes forward. Both of these bundles are vector bundles. One can also take a tensor bundle of two vector bundles by replacing the fiber over a point by the tensor product of the fibers over the same point, e.g.,

$$TM \otimes T^*M = \bigsqcup_{p \in M} \left(T_pM \otimes T_p^*M \right).$$

Note that there are canonical isomorphisms of tensor products of vector spaces, such as $V \otimes V^*$ is isomorphic to endomorphisms of V. Note the difference between bilinear forms $(V^* \otimes V^*)$, endomorphisms $(V \otimes V^*)$, and bivectors $(V \otimes V)$.

It is important to understand that these bundles are global objects, but will often be considered in coordinates. Given a coordinate $x = \left(x^i\right)$ and a point p in the coordinate patch, there is a basis $\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p$ for T_pM and dual basis $dx^1\big|_p, \ldots, dx^n\big|_p$ for T_p^*M. The generalization of the first definition above gives the idea of how one considers the trivializations of the bundle in a coordinate patch, and how the patches are linked together. Specifically, if x and y give different coordinates, for a point on the tensor bundle, one has

$$\begin{aligned} &T^{ij\cdots k}_{ab\cdots c}(x) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes \cdots \otimes \frac{\partial}{\partial x^k} \otimes dx^a \otimes dx^b \otimes \cdots \otimes dx^c \\ &= T^{ij\cdots k}_{ab\cdots c}(x(y)) \left[\frac{\partial y^\zeta}{\partial x^i} \frac{\partial y^\eta}{\partial x^j} \cdots \frac{\partial y^\theta}{\partial x^k} \frac{\partial x^a}{\partial y^\alpha} \frac{\partial x^b}{\partial y^\beta} \cdots \frac{\partial x^c}{\partial y^\gamma} \right] \frac{\partial}{\partial y^\zeta} \otimes \frac{\partial}{\partial y^\eta} \otimes \cdots \otimes \frac{\partial}{\partial y^\theta} \otimes dy^\alpha \otimes dy^\beta \otimes \cdots \otimes dy^\gamma, \end{aligned}$$

where technically everything should be at p (but as we shall see, one can consider this for all points in the neighborhood and this is considered as an equation of sections). Recall that a *section* of a bundle $\pi : E \to B$ is a function $f : B \to E$ such that $\pi \circ f$ is the identity on the base manifold B. A local section may only be defined on an open set in B. On the tangent space, sections are called *vector fields* and on the cotangent space, sections are called *forms* (or 1-forms). On a tensor bundle, sections are called *tensors*. Note that the set of $\frac{\partial}{\partial x^i}$ form a basis for the vector fields in the coordinate x, and dx^i form a basis for the local 1-forms in the coordinates. Sections in general are often written as $\Gamma(E)$ or as $C^\infty(E)$ (if we are considering smooth sections).

Now the equation above makes sense as an equation of tensors (sections of a tensor bundle). Often, a tensor will be denoted as simply

$$T^{ij\cdots k}_{ab\cdots c}.$$

Note that if we change coordinates, we have a different representation $T^{\zeta\eta\cdots\theta}_{\alpha\beta\cdots\gamma}$ of the same tensor. The two are related by

$$T^{\zeta\eta\cdots\theta}_{\alpha\beta\cdots\gamma} = T^{ij\cdots k}_{ab\cdots c}(x(y)) \left[\frac{\partial y^\zeta}{\partial x^i} \frac{\partial y^\eta}{\partial x^j} \cdots \frac{\partial y^\theta}{\partial x^k} \frac{\partial x^a}{\partial y^\alpha} \frac{\partial x^b}{\partial y^\beta} \cdots \frac{\partial x^c}{\partial y^\gamma} \right].$$

One can also take subsets or quotients of a tensor bundle. In particular, we may consider the set of symmetric 2-tensors or anti-symmetric tensors (sections of this bundle are called differential forms). In particular, we have the Riemannian metric tensor.

Definition 11 *A Riemannian metric g is a two-tensor (i.e., a section of $T^*M \otimes T^*M$) which is*

- *symmetric, i.e., $g(X,Y) = g(Y,X)$ for all $X, Y \in T_pM$, and*
- *positive definite, i.e., $g(X,X) \geq 0$ all $X \in T_pM$ and $g(X,X) = 0$ if and only if $X = 0$.*

Often, we will denote the metric as g_{ij}, which is shorthand for $g_{ij}dx^i dx^j$, where $dx^i dx^j = \frac{1}{2}\left(dx^i \otimes dx^j + dx^j \otimes dx^i\right)$. Note that if $g_{ij} = \delta_{ij}$ (the Kronecker delta) then

$$\delta_{ij}dx^i dx^j = \left(dx^1\right)^2 + \cdots (dx^n)^2 .$$

One can invariantly define a trace of an endomorphism (trace of a matrix) which is independent of the coordinate change, since

$$\begin{aligned}\sum_{a=1}^{n} T_a^a &= \sum_a T_\beta^\alpha \frac{\partial x^a}{\partial y^\alpha}\frac{\partial y^\beta}{\partial x^a} \\ &= \sum_a T_\beta^\alpha \delta_\alpha^\beta \\ &= \sum_\alpha T_\alpha^\alpha .\end{aligned}$$

In fact for any complicated tensor, one can take the trace in one up index and one down index. This is called contraction. Usually, when there is a repeated index of one up and one down, we do not write the sum. This is called Einstein summation convention. The above sum would be written

$$T_a^a = T_\alpha^\alpha .$$

It is understood that this is an equation of functions.

We cannot contract two indices up or two indices down, since this is not independent of coordinate change (try it!) However, now that we have the Riemannian metric, we can use it to "lower an index" and then trace, so we get

$$T^{ab}g_{ba} = T_a^a .$$

In order to raise the index, we need the dual to the Riemannian metric, which is g^{ab}, defined such that $g^{ab}g_{bc} = \delta_c^a$ (so g^{ab} is the inverse matrix of g_{ab}). Then we can use g^{ab} to raise indices and contract if necessary. Occasionally, extended Einstein convention is used, where all repeated indices are summed with the understanding that the Riemannian metric is used to raise or lower indices when necessary, e.g.,

$$T_{aa} = T_{ab}g^{ab} .$$

Since often we will be changing the Riemannian metric, it becomes important to understand that the metric is there when extended Einstein is used.

3.3 Connections and covariant derivatives

3.3.1 What is a connection?

A covariant derivative is a particular way of differentiating vector fields. Why do we need a new way to differentiate vector fields? Here is the idea. Suppose we want to give a notion of parallel vectors. In $\mathbb{R}^n$, we know that if we take vector fields with constant coefficients, those vectors are parallel at different points. That is, the vectors $\frac{\partial}{\partial x^1}\big|_{(0,0)}+2\,\frac{\partial}{\partial x^2}\big|_{(0,0)}$ and $\frac{\partial}{\partial x^1}\big|_{(1,-1)}+2\,\frac{\partial}{\partial x^2}\big|_{(1,-1)}$ are parallel. In fact, we could say that the vector field $\frac{\partial}{\partial x^1}+2\frac{\partial}{\partial x^2}$ is parallel since vectors at any two points are parallel. One might say it is because the coefficients of the vector field are constant (not functions of x^1 and x^2). However, this notion is not invariant under a change of coordinates. Suppose we consider the new coordinates $\left(y^1, y^2\right) = \left(x^1, \left(x^2\right)^2\right)$ away from $x^2 = 0$ (where it is not a diffeomorphism). Then the vector field in the new coordinates is

$$\frac{\partial y^i}{\partial x^1}\frac{\partial}{\partial y^i}+2\frac{\partial y^j}{\partial x^2}\frac{\partial}{\partial x^j}=\frac{\partial}{\partial y^1}+4x^2\frac{\partial}{\partial y^2}=\frac{\partial}{\partial y^1}+4\sqrt{y^2}\frac{\partial}{\partial y^2}.$$

The coefficients are not constant, but the vector field should still be parallel (we have only changed coordinates, so it is the same vector field)! So we need a notion of parallel vector field that is independent of coordinate changes (or covariant).

Remember that we want to generalize the notion that a vector field has constant coefficients. Let $X = X^i\frac{\partial}{\partial x^i}$ be a vector field in a coordinate patch. Roughly speaking, we want to generalize the notion that $\frac{\partial X^i}{\partial x^j} = 0$ for all i and j. The problem occurred because $\frac{\partial}{\partial x^1}\left(\frac{\partial}{\partial x^1}\right)$ is different in different coordinates. Thus we need to specify what this is. Certainly, since $\frac{\partial}{\partial x^i}$ is a basis, we must get a linear combination of these, so we take

$$\nabla_i\frac{\partial}{\partial x^j}=\Gamma_{ij}^k\frac{\partial}{\partial x^k}$$

for some functions Γ_{ij}^k. These symbols are called Christoffel symbols. To make sense on a vector field, we must have

$$\begin{aligned}\nabla_i\left(X\right)&=\nabla_i\left(X^j\frac{\partial}{\partial x^j}\right)\\&=\frac{\partial X^j}{\partial x^i}\frac{\partial}{\partial x^j}+X^j\Gamma_{ij}^k\frac{\partial}{\partial x^k}\\&=\left(\frac{\partial X^k}{\partial x^i}+X^j\Gamma_{ij}^k\right)\frac{\partial}{\partial x^k}.\end{aligned}$$

Notice the Leibniz rule (product rule). One can now define ∇ for any vector $Y = Y^i\frac{\partial}{\partial x^i}$ by

$$\nabla_Y X=\nabla_{Y^i\frac{\partial}{\partial x^i}}X=Y^i\left(\nabla_i X\right).$$

This action is called the covariant derivative.

One now defines Γ_{ij}^{k} in such a way that the covariant derivative transforms appropriately under change of coordinates. This gives a global object called a connection. The connection can be defined axiomatically as follows.

Definition 12 *A connection on a vector bundle $E \to B$ is a map*

$$\nabla : \Gamma(TB) \otimes \Gamma(E) \to \Gamma(E)$$

$$(X, \phi) \to \nabla_X \phi$$

satisfying:

- *Tensoriality (i.e., $C^{\infty}(B)$-linear) in the first component, i.e., $\nabla_{fX+Y}\phi = f\nabla_X \phi + \nabla_Y \phi$ for any function f and vector fields X, Y*

- *Derivation in the second component, i.e., $\nabla_X(f\phi) = X(f)\phi + f\nabla_X \phi$.*

- *$\mathbb{R}$-linear in the second component, i.e., $\nabla_X(a\phi + \psi) = a\nabla_X(\phi) + \nabla_X(\psi)$ for $a \in \mathbb{R}$.*

We will consider connections primarily on the tangent bundle and tensor bundles. Note that a connection ∇ on TM induces connections on all tensor bundles (also denoted ∇) in the following way:

- For a function f and vector field X, we define $\nabla_X f = Xf$

- For vector fields X, Y and dual form ω, we use the product rule to derive

  $$\nabla_X(\omega(Y)) = X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y)$$

  and thus

  $$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y).$$

  In particular, the Christoffel symbols for the connection on T^*M are the negative of the Christoffel symbols of TM, i.e.,

  $$\nabla_{\frac{\partial}{\partial x^i}} dx^j = -\Gamma_{ik}^{j} dx^k$$

  where Γ_{ij}^{k} are the Christoffel symbols for the connection ∇ on TM.

- For a tensor product, one defines the connection using the product rule, e.g.,

  $$\nabla_X(Y \otimes \omega) = (\nabla_X Y) \otimes \omega + Y \otimes \nabla_X \omega$$

  for vector fields X, Y and 1-form ω.

Remark 13 *The Christoffel symbols are not tensors. Note that if we change coordinates from x to $\tilde{x}$, we have*

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \nabla_{\left(\frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial}{\partial \tilde{x}^k}\right)} \left(\frac{\partial \tilde{x}^\ell}{\partial x^j} \frac{\partial}{\partial \tilde{x}^\ell} \right) = \frac{\partial \tilde{x}^\ell}{\partial x^i \partial x^j} \frac{\partial}{\partial \tilde{x}^\ell} + \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial \tilde{x}^\ell}{\partial x^j} \nabla_{\frac{\partial}{\partial \tilde{x}^k}} \frac{\partial}{\partial \tilde{x}^\ell}$$

which means that

$$\Gamma_{ij}^k = \tilde{\Gamma}_{p\ell}^m \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial \tilde{x}^\ell}{\partial x^j} \frac{\partial x^k}{\partial \tilde{x}^m} + \frac{\partial \tilde{x}^\ell}{\partial x^i \partial x^j} \frac{\partial x^k}{\partial \tilde{x}^\ell}.$$

One final comment. Recall that we motivated the connection by considering parallel vector fields. The connection gives us a way of taking a vector at a point and translating it along a curve so that the induced vector field along the curve is parallel (i.e., $\nabla_{\dot{\gamma}} X = 0$ along γ). This is called *parallel translation.*

Parallel vector fields allow one to rewrite derivatives in coordinates; that is, if $X = X^i \frac{\partial}{\partial x^i}$ is parallel, then

$$\frac{\partial X^i}{\partial x^j} = -X^k \Gamma_{jk}^i.$$

3.3.2 Torsion, compatibility with the metric, and Levi-Civita connection

There is a unique metric associated with the Riemannian metric, called the Riemannian connection or Levi-Civita connection. It satisfies two properties:

- Torsion-free (also called symmetric)
- Compatible with the metric.

Compatibility with the metric is the easy one to understand. We want the connection to behave well with respect to differentiating orthogonal vector fields. Being compatible with the metric is the same as

$$\nabla_X \left(g \left(Y, Z \right) \right) = g \left(\nabla_X Y, Z \right) + g \left(Y, \nabla_X Z \right).$$

Note that normally there would be an extra term, $\left(\nabla_X g \right) \left(Y, Z \right)$, so compatibility with the metric means that this term is zero, i.e., $\nabla g = 0$, where g is considered as a 2-tensor.

Torsion free means that the torsion tensor τ, given by

$$\tau \left(X, Y \right) = \nabla_X Y - \nabla_Y X - [X, Y]$$

vanishes. (One can check that this is a tensor by verifying that $\tau \left(fX, Y \right) = \tau \left(X, fY \right) = f \tau \left(X, Y \right)$ for any function f). It is easy to see that in coordinates, the torsion tensor is given by

$$\tau_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k,$$

which indicates why torsion-free is also called symmetric.

Tao gives a short motivation for the concept of torsion-free. Consider an infinitesimal parallelogram in the plane consisting of a point x, the flow of x along a vector field V to a point we will call $x+tV$, the flow of X along a vector field W to a point we will call $x+tW$, and then a fourth point which we will reach in two ways: (1) go to $x+tV$ and then flow along the parallel translation of W for a distance t and (2) go to $x+tW$ and then flow along the parallel translation of V for a distance t. Note that using method (1), we get that the point is

$$\begin{aligned}&\left.(x+tV+sW)\right|_{s=0}+t\left.\frac{\partial}{\partial s}\right|_{s=0}(x+tV+sW)+O\left(t^3\right)\\&=x+tV+tW+t^2\left.\frac{\partial}{\partial s}\right|_{s=0}V+O\left(t^3\right)=x+tV+tW-t^2V^iW^j\Gamma_{ji}^k\frac{\partial}{\partial x^k}+O\left(t^3\right).\end{aligned}$$

Note that using method (2), we get instead

$$x+tV+tW-t^2W^iV^j\Gamma_{ji}^k\frac{\partial}{\partial x^k}+O\left(t^3\right),$$

Thus this vector is $x+t\left(V+W\right)$ up to $O\left(t^3\right)$ only if $\Gamma_{ji}^k=\Gamma_{ij}^k$. Doing this around every infinitesimal parallelogram gives the equivalence of these two viewpoints.

Here is another:

Proposition 14 *A connection is torsion-free if and only if for any point $p\in M$, there are coordinates x around p such that $\Gamma_{ij}^k\left(p\right)=0$.*

Proof. Suppose one can always find coordinates such that $\Gamma_{ij}^k\left(p\right)=0$. Then clearly at that point, $\tau_{ij}^k=0$. However, since the torsion is a tensor, we can calculate it in any coordinate, so at each point, we have that the torsion vanishes. Now suppose the torsion tensor vanishes and let x be a coordinate around p. Consider the new coordinates

$$\tilde{x}^i\left(q\right)=x^i\left(q\right)-x^i\left(p\right)+\Gamma_{jk}^i\left(p\right)\left(x^j\left(q\right)-x^j\left(p\right)\right)\left(x^k\left(q\right)-x^k\left(p\right)\right).$$

Then notice that

$$\frac{\partial\tilde{x}^i}{\partial x^j}=\delta_j^i+\Gamma_{k\ell}^i\left(p\right)\delta_j^k\left(x^\ell-x^\ell\left(p\right)\right)+\Gamma_{k\ell}^i\left(p\right)\left(x^k-x^k\left(p\right)\right)\delta_j^\ell$$

and so

$$\frac{\partial\tilde{x}^i}{\partial x^j}\left(p\right)=\delta_j^i.$$

Thus $\tilde{x}$ is a coordinate patch in some neighborhood of p. Moreover, we have that

$$\frac{\partial^2\tilde{x}^i}{\partial x^j\partial x^k}=\Gamma_{jk}^i\left(p\right).$$

One can now verify that at p,

$$\begin{aligned}\Gamma_{ij}^{k}(p) &= \tilde{\Gamma}_{p\ell}^{m}\frac{\partial\tilde{x}^{p}}{\partial x^{i}}\frac{\partial\tilde{x}^{\ell}}{\partial x^{j}}\frac{\partial x^{k}}{\partial\tilde{x}^{m}}+\frac{\partial\tilde{x}^{\ell}}{\partial x^{i}\partial x^{j}}\frac{\partial x^{k}}{\partial\tilde{x}^{\ell}}\\ &= \tilde{\Gamma}_{ij}^{k}(p)+\Gamma_{ij}^{k}(p).\end{aligned}$$

■

The Riemannian connection is the unique connection which is both torsion-free and compatible with the metric. One can use these two properties to derive a formula for it. In coordinates, one finds that the Riemannian connection has the following Christoffel symbols

$$\Gamma_{ij}^{k}=\frac{1}{2}g^{k\ell}\left(\frac{\partial}{\partial x^{i}}g_{j\ell}+\frac{\partial}{\partial x^{j}}g_{i\ell}-\frac{\partial}{\partial x^{\ell}}g_{ij}\right).$$

One can easily verify that this connection has the properties expressed. Note that the $g_{j\ell}$ in the formula, etc. are not the tensors, but the functions. This is not a tensor equation since Γ_{ij}^{k} is not a tensor. Also note that it is very important that this is an expression in coordinates (i.e., that $\left[\frac{\partial}{\partial x^{i}},\frac{\partial}{\partial x^{j}}\right]=0$).

3.3.3 Higher derivatives of functions and tensors

One of the important reasons for having a connection is it allows us to take higher derivatives. Note that one can take the derivative of a function without a connection, and it is defined as

$$\begin{aligned}df &= \nabla f\\ df(X) &= \nabla_{X}f=X(f)\\ df &= \frac{\partial f}{\partial x^{i}}dx^{i}.\end{aligned}$$

One can also raise the index to get the gradient, which is

$$grad(f)=\nabla^{i}f\frac{\partial}{\partial x^{i}}=\frac{\partial f}{\partial x^{j}}g^{ij}\frac{\partial}{\partial x^{i}}.$$

However, to take the next derivative, one needs a connection. The second derivative, or *Hessian*, of a function is

$$\begin{aligned}Hess(f) &= \nabla^{2}f=\nabla df\\ \nabla^{2}f &= (\nabla_{i}df)\otimes dx^{i}\\ &= \left(\nabla_{i}\left(\frac{\partial f}{\partial x^{j}}dx^{j}\right)\right)\otimes dx^{i}\\ &= \left[\frac{\partial^{2}f}{\partial x^{i}\partial x^{j}}dx^{j}-\frac{\partial f}{\partial x^{j}}\Gamma_{ik}^{j}dx^{k}\right]\otimes dx^{i}\\ &= \left(\frac{\partial^{2}f}{\partial x^{i}\partial x^{j}}-\frac{\partial f}{\partial x^{k}}\Gamma_{ij}^{k}\right)dx^{j}\otimes dx^{i}.\end{aligned}$$

Often one will write the Hessian as

$$\nabla^2_{ij} f = \nabla_i \nabla_j f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k}.$$

Note that if the connection is symmetric, then the Hessian of a function is symmetric in the usual sense. The trace of the Hessian, $\triangle f = g^{ij} \nabla^2_{ij} f$, is called the Laplacian, and we will use it quite a bit.

We also may use the connection to compute acceleration of a curve. The velocity of a curve is $\dot{\gamma}$, which does not need a connection, but to compute the *acceleration*, $\nabla_{\dot{\gamma}} \dot{\gamma}$, we need the connection (one also sometimes sees the equivalent notation $D\dot{\gamma}/dt$). A curve with zero acceleration is called a *geodesic*.

Finally, given any tensor T, one can use the connection to form a new tensor ∇T, which has an extra down index.

3.4 Curvature

One can define the curvature of any connection on a bundle $E \to B$ in the following way

$$\begin{aligned} R &: \Gamma(TM) \otimes \Gamma(TM) \otimes \Gamma(E) \to \Gamma(E) \\ R(X,Y)\,\phi &= \nabla_X \nabla_Y \phi - \nabla_Y \nabla_X \phi - \nabla_{[X,Y]} \phi. \end{aligned}$$

We will consider the curvature of the Riemannian connection on the tangent bundle. One can easily see that in coordinates, the curvature is a tensor denoted as

$$\nabla_i \nabla_j \frac{\partial}{\partial x^k} - \nabla_j \nabla_i \frac{\partial}{\partial x^k} = R^\ell_{ijk} \frac{\partial}{\partial x^\ell}$$

which gives us that

$$\begin{aligned} \nabla_i \left(\Gamma^\ell_{jk} \frac{\partial}{\partial x^\ell} \right) - \nabla_j \left(\Gamma^\ell_{ik} \frac{\partial}{\partial x^\ell} \right) &= \left(\frac{\partial}{\partial x^i} \Gamma^\ell_{jk} \right) \frac{\partial}{\partial x^\ell} + \Gamma^\ell_{jk} \Gamma^m_{i\ell} \frac{\partial}{\partial x^m} - \left(\frac{\partial}{\partial x^j} \Gamma^\ell_{ik} \right) \frac{\partial}{\partial x^\ell} - \Gamma^\ell_{ik} \Gamma^m_{j\ell} \frac{\partial}{\partial x^m} \\ &= \left(\frac{\partial}{\partial x^i} \Gamma^\ell_{jk} - \frac{\partial}{\partial x^j} \Gamma^\ell_{ik} + \Gamma^m_{jk} \Gamma^\ell_{im} - \Gamma^m_{ik} \Gamma^\ell_{jm} \right) \frac{\partial}{\partial x^\ell} \end{aligned}$$

So the curvature tensor is

$$R^\ell_{ijk} = \frac{\partial}{\partial x^i} \Gamma^\ell_{jk} - \frac{\partial}{\partial x^j} \Gamma^\ell_{ik} + \Gamma^m_{jk} \Gamma^\ell_{im} - \Gamma^m_{ik} \Gamma^\ell_{jm}.$$

Often we will lower the index, and consider instead the curvature tensor

$$R_{ijk\ell} = R^m_{ijk} g_{m\ell}.$$

The Riemannian curvature tensor has the following symmetries:

- $R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij}$ (These imply that R can be viewed as a self-adjoint (symmetric) operator mapping 2-forms to 2-forms if one raises the first two or last two indices).

- (Algebraic Bianchi) $R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0.$
- (Differential Bianchi) $\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m}.$

Remark 15 *The tensor $R_{ijk\ell}$ can also be written as a tensor $R(X, Y, Z, W)$, which is a function when vector fields X, Y, Z, W are plugged in. We will sometimes refer to this tensor as* Rm. *The tensor R_{ijk}^{ℓ} is usually denoted by $R(X, Y)Z$, which is a vector field when vector fields X, Y, Z are plugged in.*

Remark 16 *Sometimes, the up index is lowered into the 3rd spot instead of the 4th, This will change the definitions of Ricci and sectional curvature below, but the sectional curvature of the sphere should always be positive and the Ricci curvature of the sphere should be positive definite.*

Remark 17 *Note that Γ_{ij}^{k} involved first derivatives of the metric, so Riemannian curvature tensor involves first and second derivatives of the metric.*

From these one can derive all the curvatures we will need:

Definition 18 *The Ricci curvature tensor R_{ij} is defined as*

$$R_{ij} = R_{\ell ij}^{\ell} = R_{\ell ijm} g^{\ell m}.$$

Note that $R_{ij} = R_{ji}$ by the symmetries of the curvature tensor. Ricci will sometimes be denoted $\mathrm{Rc}(g)$, *or* $\mathrm{Rc}(X, Y)$.

Definition 19 *The scalar curvature R is the function*

$$R = g^{ij} R_{ij}$$

Definition 20 *The sectional curvature of a plane spanned by vectors X and Y is given by*

$$K(X, Y) = \frac{R(X, Y, Y, X)}{g(X, X)\, g(Y, Y) - g(X, Y)^2}.$$

Here are some facts about the curvatures:

Proposition 21

1. *The sectional curvatures determine the entire curvature tensor, i.e., if one can calculate all sectional curvatures, then one can calculate the entire tensor.*
2. *The sectional curvature $K(X, Y)$ is the Gaussian curvature of the surface generated by geodesics in the plane spanned by X, Y.*
3. *The Ricci curvature can be written as an average of sectional curvature.*
4. *The scalar curvature can be written as an average of Ricci curvatures.*
5. *The scalar curvature essentially gives the difference between the volumes of small metric balls and the volumes of Euclidean balls of the same radius.*

6. *In 2 dimensions, each curvature determines the others.*

7. *In 3 dimensions, scalar curvature does not determine Ricci, but Ricci does determine the curvature tensor.*

8. *In dimensions larger than 3, Ricci does not determine the curvature tensor; there is an additional piece called the Weyl tensor.*

With this in mind, we can talk about several different kinds of nonnegative curvature.

Definition 22 *Let x be a point on a Riemannian manifold (M, g). Then x has*

1. *nonnegative scalar curvature if* $R(x) \geq 0$;

2. *nonnegative Ricci curvature at x if* $\mathrm{Rc}(X, X) = R_{ij} X^i X^j \geq 0$ *for every vector* $X \in T_x M$;

3. *nonnegative sectional curvature if* $R(X, Y, Y, X) = g(R(X, Y) Y, X) \geq 0$ *for all vectors* $X, Y \in T_x M$;

4. *nonnegative Riemann curvature (or nonnegative curvature operator) if* $\mathrm{Rm} \geq 0$ *as a quadratic form on* $\Omega^2(M)$, *i.e., if* $R_{ijk\ell} \omega^{ij} \omega^{k\ell} \geq 0$ *for all 2-forms* $\omega = \omega_{ij} dx^i \wedge dx^j$ *(where the raised indices are done using the metric g).*

It is not too hard to see that 4 implies 3 implies 2 implies 1. Also, in 3 dimensions, 3 and 4 are equivalent. In dimension 4 and higher, these are all distinct.

Chapter 4

Basics of geometric evolutions

4.1 Introduction

This lecture roughly follows Tao's Lecture 1. We will talk in general about flows or Riemannian metrics and Ricci flow.

We will consider a flow of Riemannian metrics to be a one-parameter family of Riemannian metrics, usually denoted $g(t)$ or $g_{ij}(t)$ or $g_{ij}(x,t)$ on a fixed Riemannian manifold M. There are more ingenious ways to define such a flow using spacetimes (called generalized Ricci flows). However, at present I do not think that they give a significant savings over the more classical idea, since one still needs to consider singular spacetimes. For more on generalized Ricci flows, consult the book by Morgan-Tian.

The family $g(t)$ is a one-parameter family of sections of a vector bundle, and one can take its derivative as

$$\frac{\partial}{\partial t} g(t) = \lim_{dt \to 0} \frac{g(t+dt) - g(t)}{dt}$$

since $g(t)$ and $g(t+dt)$ are both sections of the same vector bundle, so the difference makes sense. In fact, we can differentiate any tensor in this way. Similarly, we can try to solve differential equations of the form

$$\frac{\partial}{\partial t} g_{ij} = \dot{g}_{ij}$$

for some prescribed $\dot{g}_{ij}$. The evolution of the metric induces an evolution of the metric on the cotangent bundle, using

$$\begin{aligned} \frac{\partial}{\partial t}\left(g^{ij} g_{jk}\right) &= \frac{\partial}{\partial t} \delta^i_j \\ \frac{\partial}{\partial t} g^{ij} &= -g^{ik} \dot{g}_{k\ell} g^{\ell j}. \end{aligned}$$

The Riemannian connection is also changing if the metric is changing. Thus for a fixed vector field X, we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_i X &= \frac{\partial}{\partial t}\left[\frac{\partial X^j}{\partial x^i}\frac{\partial}{\partial x^j} + \Gamma_{ij}^k X^j \frac{\partial}{\partial x^k}\right] \\ &= X^j \dot{\Gamma}_{ij}^k \frac{\partial}{\partial x^k}.\end{aligned}$$

We can use the fact that the connection is torsion-free and compatible with the metric to derive the formula for $\dot{\Gamma}_{ij}^k$:

$$\begin{aligned}0 = \frac{\partial}{\partial t}\left(\nabla_i g_{jk}\right) &= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x^i} g_{jk} - \Gamma_{ij}^\ell g_{\ell k} - \Gamma_{ik}^\ell g_{j\ell}\right) \\ &= \nabla_i \dot{g}_{jk} - \dot{\Gamma}_{ij}^\ell g_{\ell k} - \dot{\Gamma}_{ik}^\ell g_{j\ell},\end{aligned}$$

and

$$0 = \dot{\Gamma}_{ij}^k - \dot{\Gamma}_{ji}^k$$

so we can solve for $\dot{\Gamma}_{ij}^k$ as

$$\begin{aligned}\nabla_i \dot{g}_{jk} &= \dot{\Gamma}_{ij}^\ell g_{\ell k} + \dot{\Gamma}_{ik}^\ell g_{j\ell} \\ \nabla_j \dot{g}_{ki} &= \dot{\Gamma}_{jk}^\ell g_{\ell k} + \dot{\Gamma}_{ji}^\ell g_{k\ell} \\ \nabla_k \dot{g}_{ij} &= \dot{\Gamma}_{ki}^\ell g_{\ell j} + \dot{\Gamma}_{kj}^\ell g_{i\ell}\end{aligned}$$

to get

$$\dot{\Gamma}_{ij}^k = \frac{1}{2} g^{k\ell}\left(\nabla_i \dot{g}_{j\ell} + \nabla_j \dot{g}_{i\ell} - \nabla_\ell \dot{g}_{ij}\right). \tag{4.1}$$

Remark 23 *This mimics the proof of the formula for the Riemannian connection given that it is torsion-free and compatible with the metric. There are other ways to derive this formula, for instance by computing in normal coordinates and using the fact that although Γ_{ij}^k is not a tensor, $\frac{\partial}{\partial t}\Gamma_{ij}^k$ comes from the difference of two connections and is thus a tensor. We will use this method below.*

We may now look at the induced formula for evolution of the Riemannian curvature tensor. Recall that, in coordinates,

$$R_{ijk}^\ell \frac{\partial}{\partial x^\ell} = \nabla_i\left(\Gamma_{jk}^\ell \frac{\partial}{\partial x^\ell}\right) - \nabla_j\left(\Gamma_{ik}^\ell \frac{\partial}{\partial x^\ell}\right).$$

Since we are interested in the derivative of a tensor, $\frac{\partial}{\partial t}R_{ijk}^\ell = \dot{R}_{ijk}^\ell$, we can compute this in any coordinate system we want. Recall that there is a coordinate system around p such that all Christoffel symbols vanish at p. Doing this reduces the equation to

$$\begin{aligned}\dot{R}_{ijk}^{\ell}\frac{\partial}{\partial x^{\ell}} &= \frac{\partial}{\partial t}\left[\nabla_i\left(\Gamma_{jk}^{\ell}\frac{\partial}{\partial x^{\ell}}\right)-\nabla_j\left(\Gamma_{ik}^{\ell}\frac{\partial}{\partial x^{\ell}}\right)\right]\\ &= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x^i}\Gamma_{jk}^{\ell}\frac{\partial}{\partial x^{\ell}}-\frac{\partial}{\partial x^j}\Gamma_{ik}^{\ell}\frac{\partial}{\partial x^{\ell}}\right)\\ &= \frac{\partial}{\partial x^i}\dot{\Gamma}_{jk}^{\ell}\frac{\partial}{\partial x^{\ell}}-\frac{\partial}{\partial x^j}\dot{\Gamma}_{ik}^{\ell}\frac{\partial}{\partial x^{\ell}}\\ &= \nabla_i\dot{\Gamma}_{jk}^{\ell}\frac{\partial}{\partial x^{\ell}}-\nabla_j\dot{\Gamma}_{ik}^{\ell}\frac{\partial}{\partial x^{\ell}}.\end{aligned}$$

This last piece is tensorial (recall that $\dot{\Gamma}_{ij}^{k}$ is a tensor), and thus only depends on the point, not the coordinate patch, so we must have that

$$\dot{R}_{ijk}^{\ell} = \nabla_i\dot{\Gamma}_{jk}^{\ell}-\nabla_j\dot{\Gamma}_{ik}^{\ell}.$$

We can now use the (4.1) to get

$$\begin{aligned}\dot{R}_{ijk}^{\ell} &= \nabla_i\left(\frac{1}{2}g^{\ell m}\left(\nabla_j\dot{g}_{km}+\nabla_k\dot{g}_{jm}-\nabla_m\dot{g}_{jk}\right)\right)-\nabla_j\left(\frac{1}{2}g^{\ell m}\left(\nabla_i\dot{g}_{km}+\nabla_k\dot{g}_{im}-\nabla_m\dot{g}_{ik}\right)\right)\\ &= \frac{1}{2}g^{\ell m}(\nabla_i\nabla_j\dot{g}_{km}+\nabla_i\nabla_k\dot{g}_{jm}-\nabla_i\nabla_m\dot{g}_{jk}-\nabla_j\nabla_i\dot{g}_{km}-\nabla_j\nabla_k\dot{g}_{im}+\nabla_j\nabla_m\dot{g}_{ik})\\ &= \frac{1}{2}g^{\ell m}(\nabla_i\nabla_j\dot{g}_{km}-\nabla_j\nabla_i\dot{g}_{km}+\nabla_i\nabla_k\dot{g}_{jm}-\nabla_i\nabla_m\dot{g}_{jk}-\nabla_j\nabla_k\dot{g}_{im}+\nabla_j\nabla_m\dot{g}_{ik})\\ &= \frac{1}{2}g^{\ell m}(-R_{ijk}^{p}\dot{g}_{mp}-R_{ijm}^{p}\dot{g}_{kp}+\nabla_i\nabla_k\dot{g}_{jm}-\nabla_i\nabla_m\dot{g}_{jk}-\nabla_j\nabla_k\dot{g}_{im}+\nabla_j\nabla_m\dot{g}_{ik}).\end{aligned}$$

We can take the trace $\dot{R}_{jk}=\dot{R}_{ijk}^{i}$ to get

$$\begin{aligned}\dot{R}_{jk} &= \frac{1}{2}g^{im}(-R_{ijk}^{p}\dot{g}_{mp}-R_{ijm}^{p}\dot{g}_{kp}+\nabla_i\nabla_k\dot{g}_{jm}-\nabla_i\nabla_m\dot{g}_{jk}-\nabla_j\nabla_k\dot{g}_{im}+\nabla_j\nabla_m\dot{g}_{ik})\\ &= -\frac{1}{2}g^{im}R_{ijk}^{p}\dot{g}_{mp}+\frac{1}{2}g^{pq}R_{jp}\dot{g}_{kp}-\frac{1}{2}g^{im}\nabla_i\nabla_m\dot{g}_{jk}-\frac{1}{2}g^{im}\nabla_j\nabla_k\dot{g}_{im}\\ &\quad+\frac{1}{2}g^{im}(\nabla_k\nabla_i\dot{g}_{jm}-R_{ikj}^{p}\dot{g}_{pm}-R_{ikm}^{p}\dot{g}_{jp}+\nabla_j\nabla_m\dot{g}_{ik})\\ &= -g^{im}R_{ijk}^{p}\dot{g}_{mp}+\frac{1}{2}g^{pq}R_{jp}\dot{g}_{kp}+\frac{1}{2}g^{pq}R_{kq}\dot{g}_{jp}-\frac{1}{2}g^{im}\nabla_i\nabla_m\dot{g}_{jk}\\ &\quad-\frac{1}{2}g^{im}\nabla_j\nabla_k\dot{g}_{im}+\frac{1}{2}g^{im}(\nabla_k\nabla_i\dot{g}_{jm}+\nabla_j\nabla_m\dot{g}_{ik})\\ &= -\frac{1}{2}\triangle_L\dot{g}_{jk}-\frac{1}{2}g^{im}\nabla_j\nabla_k\dot{g}_{im}+\frac{1}{2}g^{im}(\nabla_k\nabla_i\dot{g}_{jm}+\nabla_j\nabla_m\dot{g}_{ik})\end{aligned}$$

where

$$\triangle_L\dot{g}_{jk} = g^{im}\nabla_i\nabla_m\dot{g}_{jk}+2g^{im}R_{ijk}^{p}\dot{g}_{mp}-g^{pq}R_{jp}\dot{g}_{kp}-g^{pq}R_{kq}\dot{g}_{jp}$$

is the Lichnerowitz Laplacian (notice only the first term has two derivatives of $\dot{g}_{jk}$). (Note: I think that T. Tao has an error in this formula with the sign of the last term of $\dot{R}_{jk}$.

Finally, we may take a trace to get

$$\begin{aligned}\dot{R} &= g^{jk}\dot{R}_{jk} - g^{pj}\dot{g}_{pq}g^{qk}R_{jk} \\ &= -g^{pq}g^{jk}R_{jp}\dot{g}_{kp} - g^{im}\nabla_i\nabla_m g^{jk}\dot{g}_{jk} \\ &\quad + g^{im}g^{jk}\nabla_k\nabla_i\dot{g}_{jm} \\ &= -\langle \mathrm{Rc}, \dot{g}\rangle - \triangle \operatorname{tr}_g(\dot{g}) + \operatorname{div}\operatorname{div}\dot{g}\end{aligned}$$

where

$$(\operatorname{div} h)_j = g^{k\ell}\nabla_k h_{\ell j}$$

4.2 Ricci flow

Note that in the evolution of Ricci curvature, if one considers

$$\dot{g} = -2\,\mathrm{Rc},$$

one gets

$$\begin{aligned}&-\frac{1}{2}g^{im}\nabla_j\nabla_k\dot{g}_{im} + \frac{1}{2}g^{im}(\nabla_k\nabla_i\dot{g}_{jm} + \nabla_j\nabla_m\dot{g}_{ik}) \\ &= g^{im}\nabla_j\nabla_k R_{im} - g^{im}\nabla_k\nabla_i R_{jm} - g^{im}\nabla_j\nabla_m R_{ik} \\ &= \nabla_j\nabla_k R - g^{im}\nabla_j\nabla_m R_{ik} - g^{im}\nabla_k\nabla_i R_{jm}\end{aligned}$$

Note that the differential Bianchi identity implies

$$\begin{aligned}0 &= g^{jm}g^{k\ell}\left(\nabla_i R_{jk\ell m} + \nabla_j R_{ki\ell m} + \nabla_k R_{ij\ell m}\right) \\ &= \nabla_i R - g^{jm}\nabla_j R_{im} - g^{k\ell}\nabla_k R_{ij\ell m}\end{aligned}$$

so

$$\nabla_i R = 2g^{jm}\nabla_j R_{im}$$

so

$$-\frac{1}{2}g^{im}\nabla_j\nabla_k\dot{g}_{im}+\frac{1}{2}g^{im}(\nabla_k\nabla_i\dot{g}_{jm}+\nabla_j\nabla_m\dot{g}_{ik}) = \nabla_j\nabla_k R-\frac{1}{2}\nabla_j\nabla_k R-\frac{1}{2}\nabla_k\nabla_j R = 0.$$

Thus under Ricci flow,

$$\frac{\partial}{\partial t}\mathrm{Rc} = \triangle_L \mathrm{Rc}\,.$$

Furthermore, We see that

$$\begin{aligned}\frac{\partial R}{\partial t} &= 2\langle \mathrm{Rc}, \mathrm{Rc}\rangle + 2\triangle R - 2g^{i\ell}g^{jk}\nabla_i\nabla_j R_{k\ell} \\ &= -2\langle \mathrm{Rc}, \mathrm{Rc}\rangle + 2\triangle R - g^{i\ell}\nabla_i\nabla_\ell R \\ &= \triangle R + 2\left|\mathrm{Rc}\right|^2\end{aligned}$$

The important notion to get right now is that this looks very much like a heat equation with a reaction term. We will see how to make use of this in the near future.

4.3 Existence/Uniqueness

Note that the Ricci flow equation,

$$\frac{\partial}{\partial t}g = -2\,\mathrm{Rc}$$

is a second order partial differential equation, since the Ricci curvature comes from second derivatives of the metric. To truly look at existence/uniqueness, one must write this as an equation in coordinates. We will look at the linearization of this operator in order to find the principle symbol (which is basically the coefficients of the linearization of the highest derivatives). Analysis of the principle symbol will often allow us to determine that a solution exists for a short time. Here is the meta-theorem for existence of parabolic PDE:

Meta-Theorem (imprecise): *A semi-linear PDE of the form*

$$\frac{\partial u}{\partial t} - a^{ij}\left(x,t\right)\frac{\partial^2 u}{\partial x^i \partial x^j} + F\left(x,t,u,\partial u\right) = 0.$$

on a compact manifold has a solution with initial condition $u\left(x,0\right) = f\left(x\right)$ *if there exists* $\delta > 0$ *such that* $a^{ij}\xi_i\xi_j \geq \delta\left|\zeta\right|^2$ *(this condition is called strict parabolicity) for* t *close to 0. Similarly, if we allow* a^{ij} *to depend on* u *(making the equation quasilinear, the same is true if we look at the linearization (which is then semilinear).*

Remark 24 a^{ij} *is called the principal symbol of the parabolic differential operator. If one takes out the* $\frac{\partial}{\partial t}$*, the differential operator is said to be elliptic if it satisfies the inequality.*

Remark 25 *We can replace* $\frac{\partial}{\partial x^i}$ *with* ∇_i *since the difference has fewer derivatives.*

Remark 26 *One should be able to prove a coordinate independent version, but this is not usually done. All theory is based on theory of differential equations on domains in the plane.*

Remark 27 *For an arbitrary, nonlinear second order PDE of the form*

$$G\left(x,t,u,\partial u,\partial^2 u\right) = 0,$$

one can consider the linearization of G *with respect to* u*. This will look roughly like*

$$\left[\partial_{H_{ij}}G\frac{\partial^2}{\partial x^i \partial x^j} + \partial_{V_i}G\frac{\partial}{\partial x^i} + \partial_u G\right]v$$

where $G = G\left(x,t,u,V,H\right)$ *and the operator is evaluated at some* u *(which is where it has been linearized. Notice this now gives a semilinear PDE. The principle symbol is* $\partial_{H_{ij}}G\xi_i\xi_j$*.*

Example 28 *Note that the equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\left(\partial x^1\right)^2} u + \frac{\partial^2}{\left(\partial x^2\right)^2} u = \delta^{ij} \frac{\partial^2}{\partial x^i \partial x^j}$$

is already linear. For an equation like

$$\frac{\partial u}{\partial t} = u^2 \frac{\partial^2 u}{\partial x^2},$$

the linearization is

$$\frac{\partial v}{\partial t} = u^2 \frac{\partial^2 v}{\partial x^2} + 2u \frac{\partial^2 u}{\partial x^2} v,$$

thus the principal symbol is u^2 which is positive if $u > 0$.

Now, the Ricci operator is an operator on sections, not just functions, so how do we make sense of the kind of result given above. We can make a similar definition in terms of the linearization, but now the principle symbol is a map from sections of the symmetric 2-tensor bundle to itself. What we need is that for any $\xi \neq 0$, the principle symbol is a linear isomorphism.

Recall that the linearization of R_{jk} is

$$\dot{R}_{jk} = -\frac{1}{2} g^{im} \nabla_i \nabla_m \dot{g}_{jk} - \frac{1}{2} g^{im} \nabla_j \nabla_k \dot{g}_{im} + \frac{1}{2} g^{im} (\nabla_k \nabla_i \dot{g}_{jm} + \nabla_j \nabla_m \dot{g}_{ik})$$

so the principle symbol of $-2R_{jk}$ is

$$\hat{\sigma} \left[D\,\mathrm{Rc} \right] (\xi) \, (h) = g^{im} \xi_i \xi_m h_{jk} + g^{im} \xi_j \xi_k h_{im} - g^{im} (\xi_k \xi_i h_{jm} + \xi_j \xi_k h_{ik}).$$

In order to see if this is an isomorphism, we can rotate ξ so that $\xi_1 > 0$ and $\xi_2 = \cdots = \xi_n = 0$ and by scaling we can assume $\xi_1 = 1$. We can also assume that at a point $g_{ij} = \delta_{ij}$. Then we see that

$$\hat{\sigma} \left[D\,\mathrm{Rc} \right] (\xi) \, (h)_{jk} = h_{jk} + \delta^1_j \delta^1_k \left(h_{11} + \cdots + h_{nn} \right) - (\delta^1_k h_{j1} + \delta^1_j h_{1k}).$$

And so the matrix for the symbol gives

$$\left(\begin{array}{c|ccc} h_{22} + \cdots + h_{nn} & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & h_{\alpha\beta} & \\ 0 & & & \end{array} \right)$$

where $2 \leq \alpha, \beta \leq n$. We see immediately that there is an n-dimensional kernel (we can let h_{1k} equal anything we want and if everything else is zero, we are in the kernel).

Now we will see how to overcome this issue. Rewrite the linearization as

$$\begin{aligned} \dot{R}_{jk} &= -\frac{1}{2} g^{im} \nabla_i \nabla_m \dot{g}_{jk} - \frac{1}{2} g^{im} \nabla_j \nabla_k \dot{g}_{im} + \frac{1}{2} g^{im} (\nabla_k \nabla_i \dot{g}_{jm} + \nabla_j \nabla_m \dot{g}_{ik}) \\ &= -\frac{1}{2} g^{im} \nabla_i \nabla_m \dot{g}_{jk} + \frac{1}{2} \nabla_k V_j + \nabla_j V_k \end{aligned}$$

if

$$V_j = g^{im} \nabla_i \dot{g}_{jm} - \frac{1}{2} \nabla_j \left(g^{im} \dot{g}_{im} \right).$$

The last term is equal to the Lie derivative $L_V g_{jk}$ (where $V = V^i = g^{ij} V_j$, often denoted $V^{\#}$) and so we get that the linearization of $-2R_{jk}$ is

$$g^{im} \nabla_i \nabla_m \dot{g}_{jk} - L_V g_{jk}.$$

Lie derivatives arise from changing by diffeomorphisms, i.e., if ϕ_t are diffeomorphisms such that

$$\frac{d}{dt} \phi_t (x) = X (x)$$

and ϕ_0 is the identity (i.e., ϕ_t is the flow of X), then

$$\left. \frac{d}{dt} \right|_{t=0} \phi_t^* g_0 = L_X g_0.$$

One can pretty easily see that if we take the vector field V as above, we can look at the flow ϕ_t and $\phi_t^* g (t)$ and we will see that $\tilde{g} (t) = \phi_t^* g (t)$ evolves by

$$\frac{\partial}{\partial t} \tilde{g} = -2 \operatorname{Rc} (\tilde{g}) + L_V \tilde{g}$$

and the linearization is

$$g^{im} \nabla_i \nabla_m \dot{g}_{jk}.$$

This is like looking at an equation roughly like

$$\frac{\partial}{\partial t} h = g^{im} \nabla_i \nabla_m h_{jk},$$

which is a heat equation with a unique solution. This has principal symbol g^{ij}, which is strictly positive definite. It can be shown that this implies that the modified Ricci flow (the equation above on $\tilde{g}$) has a unique solution. One can then show that this implies the Ricci flow has a unique solution too. I.e.,

Theorem 29 *Given an initial closed Riemannian manifold (M, g_0), there is a time $T > 0$ and Riemannian metrics $g (t)$ on M for each $t \in [0, T)$ such that which satisfy the initial value problem*

$$\frac{\partial}{\partial t} g = -2 \operatorname{Rc} (g)$$
$$g (0) = g_0.$$

Moreover, given the initial condition, $g (t)$ are uniquely determined, and there is a maximal such T.

Remark 30 *One can also show that this is true for complete manifolds with bounded curvature $|\operatorname{Rm}|$, which was done by Shi. However, the proof is much more difficult on noncompact manifolds.*

Chapter 5

Basics of PDE techniques

5.1 Introduction

This section will roughly follow Tao's lecture 3. We will look at some basic PDE techniques and apply them to the Ricci flow to obtain some important results about preservation and pinching of curvature quantities. The important fact is that the curvatures satisfy certain reaction-diffusion equations which can be studied with the maximum principle.

5.2 The maximum principle

Recall that if a smooth function $u : U \to \mathbb{R}$ where $U \subset \mathbb{R}^n$ has a local minimum at x_0 in the interior of U, then

$$\frac{\partial u}{\partial x^i}(x_0) = 0$$
$$\frac{\partial^2 u}{\partial x^i \partial x^j}(x_0) \geq 0$$

where the second statement is that the Hessian is nonnegative definite (has all nonnegative eigenvalues). The same is true on a Riemannian manifold, replacing regular derivatives with covariant derivatives.

Lemma 31 *Let (M, g) be a Riemannian manifold and $u : M \to \mathbb{R}$ be a smooth (or at least C^2) function that has a local minimum at $x_0 \in M$. Then*

$$\nabla_i u(x_0) = 0$$
$$\nabla_i \nabla_j u(x_0) \geq 0$$
$$\triangle u(x_0) = g^{ij}(x_0) \nabla_i \nabla_j u(x_0) \geq 0.$$

Proof. In a coordinate patch, the first statement is clear since $\nabla_i u = \frac{\partial u}{\partial x^i}$. The second statement is that the Hessian is positive definite. Recall that in coordinates, the Hessian is

$$\nabla_i \nabla_j u = \frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u}{\partial x^k},$$

but at a minimum, the second term is zero and the positive definiteness follows from the case in $\mathbb{R}^n$. The last statement is true since both g and the Hessian are positive definite. ■

Remark 32 *There is a similar statement for maxima.*

The following lemma is true in the generality of a smooth family of metrics, though is also of use for a fixed metric.

Lemma 33 *Let $(M, g(t))$ be a smooth family of compact Riemannian manifolds for $t \in [0,T]$. Let $u : [0,T] \times M \to \mathbb{R}$ be a C^2 function such that*

$$u(0,x) \geq 0$$

for all $x \in M$. Also let $A \in \mathbb{R}$. Then exactly one of the following is true:

1. *$u(t,x) \geq 0$ for all $(t,x) \in [0,T] \times M$, or*
2. *There exists a $(t_0, x_0) \in (0,T]$ such that all of the following are true:*

$$\begin{aligned} u(t_0,x_0) &< 0 \\ \nabla_i u(t_0,x_0) &= 0, \\ \triangle_{g(t_0)} u(t_0,x_0) &\geq 0, \\ \frac{\partial u}{\partial t}(x_0,t_0) &< 0. \end{aligned}$$

Proof. Consider the function

$$u(t,x) + \varepsilon t.$$

If $u(t,x) + \varepsilon t > 0$ for all $\varepsilon > 0$ (small), then $u(t,x) \geq 0$. Otherwise there is an $\varepsilon > 0$ and an initial t_0 such that there is a $x_0 \in M$ such that $u(t_0,x_0) + \varepsilon t_0 = 0$. At the first such time, we must have that x_0 is a spatial minimum for this function, and thus

$$\begin{aligned} u(t_0,x_0) &= -\varepsilon t_0 < 0 \\ \nabla u(t_0,x_0) &= 0 \\ \triangle u(t_0,x_0) &\geq 0 \\ \frac{\partial u}{\partial t}(t_0,x_0) &\leq -\varepsilon < 0. \end{aligned}$$

■

Corollary 34 *Let $(M, g(t))$ be a smooth family of compact Riemannian manifolds for $t \in [0, T]$. Let $u, v : [0, T] \times M \to \mathbb{R}$ be C^2 functions such that*

$$u(0, x) \geq v(0, x)$$

for all $x \in M$. Also let $A \in \mathbb{R}$. Then exactly one of the following is true:

1. *$u(t, x) \geq v(t, x)$ for all $(t, x) \in [0, T] \times M$, or*
2. *There exists a $(t_0, x_0) \in (0, T]$ such that all of the following are true:*

$$\begin{aligned} u(t_0, x_0) &< v(t_0, x_0) \\ \nabla_i u(t_0, x_0) &= \nabla_i v(t_0, x_0), \\ \triangle_{g(t_0)} u(t_0, x_0) &\geq \triangle_{g(t_0)} v(t_0, x_0), \\ \frac{\partial u}{\partial t}(x_0, t_0) &< \frac{\partial v}{\partial t}(t_0, x_0) + A[u(t_0, x_0) - v(t_0, x_0)]. \end{aligned}$$

Proof. Replace u with $e^{-At}(u - v)$. ■

This will allow us to estimate subsolutions of a heat equation by supersolutions of the same heat equation.

Corollary 35 *Let the assumptions be the same as in Corollary 34, including*

$$u(0, x) \geq v(0, x).$$

Suppose u is a supersolution of a reaction-diffusion equation, i.e.,

$$\frac{\partial u}{\partial t} \geq \triangle_{g(t)} u + \nabla_{X(t)} u + F(t, u)$$

and v is a subsolution of the same equation, i.e.,

$$\frac{\partial v}{\partial t} \leq \triangle_{g(t)} u + \nabla_{X(t)} v + F(t, v)$$

for all $(t, x) \in [0, T] \times M$, where $X(t)$ is a vector field for each t and $F(t, w)$ is Lipschitz in w, i.e., there is $A > 0$ such that

$$|F(t, w) - F(t, w')| \leq A|w - w'|.$$

Then

$$u(t, x) \geq v(t, x)$$

for all $t \in [0, T]$.

Proof. Consider

$$\begin{aligned} \frac{\partial}{\partial t}(u - v) &\geq \triangle_{g(t)}(u - v) + \nabla_{X(t)}(u - v) + F(t, u) - F(t, v) \\ &\geq \triangle_{g(t)}(u - v) + \nabla_{X(t)}(u - v) - A|u - v|. \end{aligned}$$

The dichotomy in Corollary 34 says that either $u-v \geq 0$ for all t, x or else there is a point (t_0, x_0) such that at that point,

$$\begin{aligned} u - v &< 0 \\ \triangle (u-v) &= 0 \\ \nabla (u-v) &= 0 \\ \frac{\partial}{\partial t}(u-v) &\leq A'(u-v) = -A'|u-v| \end{aligned}$$

for any A'. But the inequality above says that at that same point

$$\frac{\partial}{\partial t}(u-v) \geq -A|u-v|,$$

which is a contradiction if $-A' < -A$. ■

Usually, instead of making v a subsolution, we will just make v the subsolution to the ODE

$$\frac{dv}{dt} \leq F(t,v),$$

where $v = v(t)$ is independent of x and so this is also a subsolution to the PDE.

Here is an easy application:

Proposition 36 *Nonnegative scalar curvature is preserved by the Ricci flow, i.e., if $R(0,x) \geq 0$ for all $x \in M$ and the metric g satisfies the Ricci flow for $t \in [0,T)$, then $R(t,x) \geq 0$ for all $x \in M$ and $t \in [0,T]$.*

Proof. Recall that R satisfies the evolution equation

$$\frac{\partial R}{\partial t} = \triangle_{g(t)} R + 2|\mathrm{Rc}|^2,$$

thus it is a supersolution to the heat equation (with changing metric), i.e.,

$$\frac{\partial R}{\partial t} \geq \triangle R.$$

By Corollary 35, we must have that $R \geq 0$ for all t. ■

We can actually do better. Notice that if T_{ij} is a 2-tensor on an n-dimensional Riemannian manifold (M,g), then

$$|T|^2 \geq \frac{1}{n}\left(g^{ij}T_{ij}\right)^2$$

since

$$\left| T_{ij} - \frac{1}{n}\left(g^{k\ell}T_{k\ell}\right) g_{ij} \right|^2 \geq 0$$

(expand that out and see it implies the previous inequality). Thus

$$|\mathrm{Rc}|^2 \geq \frac{1}{n} R^2$$

and so scalar curvature satisfies

$$\frac{\partial R}{\partial t} \geq \triangle R + \frac{2}{n} R^2.$$

The maximum principle implies that $R(t,x) \geq f(t)$ for all $x \in M$, where $f(t)$ is the solution to the ODE

$$\begin{aligned} \frac{df}{dt} &= \frac{2}{n} f^2 \\ f(0) &= \min_{x \in M} R(x,0). \end{aligned}$$

This equation can be solved explicitly as

$$\begin{aligned} \int \frac{1}{f^2} df &= \int \frac{2}{n} dt \\ -\frac{1}{f} &= \frac{2}{n} t - \frac{1}{f(0)} \\ f(t) &= \frac{f(0)}{1 - \frac{2}{n} f(0) t} \end{aligned}$$

as long as $f(0) \neq 0$. Notice that if $f(0) > 0$ then this says that $R(t,x)$ goes to infinity in finite time $T \leq \frac{n}{2f(0)}$. If $f(0) < 0$, then this says that if the flow exists for all time, then the scalar curvature becomes nonnegative in the limit.

5.3 Maximum principle on tensors

Sometimes it may be useful to use a tensor variant, for a function $u : [0,T] \to \Gamma(V)$ where $\Gamma(V)$ are sections of a tensor bundle (such as if we wish to apply the maximum principle to the Ricci tensor, for instance). Here is the theorem (possibly due to Hamilton?)

Lemma 37 *Let (M,g) be a d-dimensional Riemannian manifold and let V be a vector bundle over M with connection ∇. Let K be a closed, fiberwise convex subset of V which is parallel with respect to the connection. Let $u \in \Gamma(V)$ be a section such that*

1. *$u(x) \in \partial K_x$ at some point $x \in M$, and*
2. *$u(y) \in K_y$ for all y in a neighborhood of x*

(This is the notion that u attains a maximum at x.) Then $\nabla_X u(x)$ is tangent to K_x at $u(x)$ and the Laplacian $\triangle u(x) = g^{ij}(x) \nabla_i \nabla_j u(x)$ is an inward or tangential pointing vector to K_x at $u(x)$.

Here are the relevant definitions.

Definition 38 *A subset K of a tensor bundle $\pi : E \to M$ is fiberwise convex if the fiber $K_x = K \cap E_x$ (where $E_x = \pi^{-1}(x)$) is a convex subset of the vector space E_x.*

Definition 39 *A subset K is parallel to the connection ∇ if it is preserved by parallel translation, i.e., if $P_{x,y}$ is parallel translation along a curve from x to y, then $P^*_{x,y} K_y \subset K_x$ (this is if the tensors are all contravariant).*

Example 40 *The set of positive definite two-tensors is fiberwise convex and parallel with respect to the Levi-Civita connection.*

The maximum principle on tensors can be used to show things like:

1. Nonnegative Ricci curvature is preserved by Ricci flow in dimension 3.
2. Nonnegative curvature operator is preserved by Ricci flow in all dimensions.

We will go into this in more detail in future lectures.

Remark 41 *Why do we need convex? Consider the scalar case where we replace $u(t,x) \geq 0$ with $u(t,x) \geq 1 - x^2$ nearby $x = 0$. Consider the first time $u(t,x)$*

Chapter 6

Singularities of Ricci Flow

6.1 Introduction

This roughly covers Lecture 7 of Tao. We will look at analysis of singularities of Ricci flow.

6.2 Finite time singularities

Recall that we proved short time existence for Ricci flow, which means the flow exists until it reaches a singularity. The first, most basic result on singularities is the following result of Hamilton:

Theorem 42 *Let $(M, g(t))$ be a solution to the Ricci flow on a compact manifold on a maximal time interval $[0, T)$. If $T < \infty$, then*

$$\lim_{t \to T^-} \max_{x \in M} |\mathrm{Rm}(t)|^2_{g(t)} = \infty.$$

Remark 43 *Uniqueness of the Ricci flow is necessary for there to exist a maximal time interval. The idea is that if there is a solution on $[0, T_1)$ and another solution on $[T_1 - \varepsilon, T_2)$, then they must agree on the overlap, so one can consider the flow on $[0, T_2)$ which extends both the flows. Now simply take the sup of all T_2 for which the flow exists and this forms the maximal time interval.*

Remark 44 *N. Sesum was able to replace $|\mathrm{Rm}|$ with $|\mathrm{Rc}|$.*

Proof (idea). Suppose $T < \infty$ and $|\mathrm{Rm}|$ remains bounded. Then one can show that all derivatives $\left|\nabla^k \mathrm{Rm}\right|$ are uniformly bounded for $t \in [\varepsilon, T)$. One can use this to derive uniform bounds on the metric and its derivatives and then to extract a smooth limit metric at T (of a subsequence using an Arzela-Ascoli type compactness theorem). The existence/uniqueness result tells us that we can extend the flow for a short time, contradicting the fact that $[0, T)$ was maximal. ∎

This is quite a useful theorem, but it is still possible to develop singularities in a finite time (for instance, the sphere or a neck pinch). It will be very important to analyze what is happening at the singular times so that we can do surgery to remove the problems and then continue the flow. We will do this using a PDE technique called blowing up around the singularity, which uses scaling to allow us to see the precise behavior of the flow near the singular time.

6.3 Blow ups

Here is the idea. At the singular time, we know that $|\mathrm{Rm}|^2$ is going to infinity. If we scale the metric g to cg, then we get

$$|\mathrm{Rm}\,(cg)|^2 = \frac{1}{c^2}\,|\mathrm{Rm}\,(g)|^2$$

or

$$|\mathrm{Rm}\,(cg)| = \frac{1}{c}\,|\mathrm{Rm}\,(g)|\,,$$

so if we want to prevent $|\mathrm{Rm}|$ from going to infinity, we choose a scaling

$$L_n^{-2} = \max_{x\in M} |\mathrm{Rm}\,(g\,(t_n))|$$

and rescale the metric $g\,(t_n)$ by L_n^{-2}. We can actually rescale to a sequence of solutions of the differential equation (Ricci flow) by looking at

$$g_n\,(t) = \frac{1}{L_n^2}\,g\left(t_n + tL_n^2\right),$$

where $t_n \to T$, the singular time. Notice that

$$\begin{aligned} \frac{\partial}{\partial t} g_n &= \frac{\partial}{\partial t}\left[L_n^{-2} g\left(t_n + tL_n^2\right)\right] \\ &= -2\,\mathrm{Rc}\left[g\left(t_n + tL_n^2\right)\right] \\ &= -2\,\mathrm{Rc}\,[g_n\,(t)] \end{aligned}$$

so g_n is a sequence of solutions to the Ricci flow whose initial value is getting closer to the singular time. Furthermore, the initial curvatures $|\mathrm{Rm}\,(g_n\,(0))|$ are all bounded by 1. On the down side, $L_n \to 0$ and so the metric is being multiplied by larger and larger scaling factors and thus it is quite likely that a limit will become noncompact. We will need to have a good notion of convergence which allows convergence to noncompact manifolds.

Remark 45 *We do not necessarily have to choose L_n as above. The fact that $L_n^{-2} \to \infty$ is why this is called a* blow-up. *If we take $L_n^{-2} \to 0$ then we have what is called a* blow-down.

Notice that if the original Ricci flow is defined on an interval $[0, T)$, then the rescaled solutions $g_n(t)$ exist on the interval

$$\left[-\frac{t_n}{L_n^2}, \frac{T-t_n}{L_n^2}\right).$$

Thus if we can extract a limit and $t_n \to T$ and $L_n \to 0$, then the limit metrics will be ancient, i.e., will start at $t = -\infty$ (the final endpoint depends a little more on how we choose the t_n with respect to how we pick the L_n, allowing it to be 0, positive, or $+\infty$; we may have reason to choose any of these).

The main goal is to find quantities which

1. become better as we go to the limit (Tao calls these critical or subcritical) and
2. severely restrict the geometry of the limit (Tao calls these coercive).

Examples of quantities to study or not:

- The volume of the manifold. If we look at
$$V(M, g) = \int_M dV_g,$$
we see that
$$V\left(M, \frac{1}{L_n^2} g\right) = \frac{1}{L_n^{2/d}} V(M, g)$$
and so if $L_n \to 0$, this quantity does not persist to the limit. Tao calls this behavior supercritical.

- The total scalar curvature. If we look at
$$F(M, g) = \int_M R dV,$$
we see that
$$F\left(M, \frac{1}{L_n^2} g\right) = L_n^{2-d} F(M, g).$$
Thus it is preserved in dimension 2 (this is the Gauss-Bonnet theorem, which implies critical behavior) and does not persist in higher dimensions (supercritical).

- The minimum scalar curvature. If we look at
$$R_{\min}(M, g) = \min_{x \in M} R(x),$$
then we see that
$$R_{\min}\left(M, \frac{1}{L_n^2} g\right) = L_n^2 R_{\min}(M, g)$$
and so this goes to zero as $L_n \to 0$. This is subcritical behavior. Unfortunately, $R_{\min} = 0$ does not give sufficient information of the limit to classify (not coercive enough).

- Lowest eigenvalue of the operator $-4\triangle u+R$. Notice that this is subcritical since if we call this functional $H(M,g)$

$$H\left(M,\frac{1}{L_n^2}g\right)=L^2H(M,g).$$

It turns out that this quantity obeys a monotonicity and has some nice coercivity, though not quite enough. We will soon look at a particular critical (i.e., scale invariant) quantity which is related.

6.4 Convergence and collapsing

Manifolds may converge in a number of ways. Here are some examples:

- Sphere converging to a point
- Sphere converging to a cylinder
- Cylinder collapsing to a line
- Torus collapsing to a circle
- Torus collapsing to a line
- 3-sphere collapsing to a 2-sphere by shrinking Hopf fibers

There are several issues here, notably:

- Is there collapse?
- Does convergence involve noncompact manifolds?

The most obvious notion of collapse involves the injectivity radius going to zero. Recall that the exponential map is the map from the tangent space at one point to the manifold where a vector v is taken to the point one unit from the origin along a geodesic starting with velocity v. This map is a local diffeomorphism, and there is an $r>0$ such that the ball $B(0,r)$ is mapped diffeomorphically to a ball on the manifold. The largest such r is called the *injectivity radius*. As this goes to zero, there is collapse.

We will see another way to measure this collapse soon.

For noncompact manifolds, one needs to consider pointed convergence. This generally involves looking at convergence of balls of larger and larger size. If all manifolds and their limit have a uniform diameter bound, then one does not need to consider pointed convergence.

6.5 κ-noncollapse

One definition of a collapsing sequence is the following:

Definition 46 *A pointed sequence* (M_n, g_n, p_n) *of Riemannian manifolds is collapsing if* $\mathrm{inj}_{p_n} \to 0$ *as* $n \to \infty$.

We may wish to rescale the manifolds (M_n, g_n) to be of some uniform size, say by making $|\mathrm{Rm}\,(p_n)| = 1$. Then one can consider a rescaled collapsing if $|\mathrm{Rm}\,(p_n)|^{1/2}\,\mathrm{inj}_{p_n} \to 0$.

When one assumes that the sectional curvature is bounded, then the collapse is restricted. Let $V(U) = V_g(U)$ denote the Riemannian volume of the Borel subset $U \subset M$.

Theorem 47 (Cheeger) *Suppose that* $|\mathrm{Rm}|_g \leq C r_0^{-2}$ *on* $B(p, r_0) \subset M^d$ *and that*

$$V(B(p, r_0)) \geq \delta r_0^d$$

for some $\delta > 0$. *Then the injectivity radius of* p, inj_p, *is at least*

$$\mathrm{inj}_p \geq c r_0$$

for some constant $c = c(C, \delta, d) > 0$.

Let's think about this theorem for a minute, to see if it is ever applicable since it seems the assumptions are quite strong. Note that as $r_0 \to 0$, the ball looks more and more Euclidean. That means that for very small $r_0 \ll \mathrm{inj}_p$,

$$V(B(p, r_0)) \approx \omega r^d,$$

where $\omega = \omega(d)$ is the correct constant for a Euclidean ball, and

$$|\mathrm{Rm}|_g \approx |\mathrm{Rm}\,(p)|_g\,.$$

So as $r_0 \to 0$, we see that

$$\lim_{r_0 \to 0} \left(r_0^2 \sup_{x \in B(p, r_0)} |\mathrm{Rm}\,(x)| \right) = 0$$

$$\lim_{r_0 \to 0} \frac{V(B(p, r_0))}{\omega r^d} = 1.$$

In particular, for any $C > 0$ and $0 < \delta < 1$, there is a $r_* > 0$ such that the assumptions are satisfied if $r_0 \leq r_*$.

Note that the converse is also true from more classical results.

Theorem 48 *If* $|\mathrm{Rm}|_g \leq C$ *and* $\mathrm{inj}_p \geq \iota$, *then there is a* $\delta = \delta(C, \iota, d)$ *such that*

$$V(B(p, r)) \geq \delta r^d$$

for all $r \leq \iota$.

Collapse generally refers to the injectivity radius going to zero. Cheeger's theorem tells us that when curvature is bounded, volume of balls getting small and injectivity radius getting small are essentially the same. This roughly motivates the following noncollapsing definition, which is not quite the definition we will use.

Definition 49 *A Riemannian manifold (M^d, g) is κ-collapsed at $p \in M$ at scale r_0 if*

1. *(Bounded normalized curvature)* $|\mathrm{Rm}|_g \leq r_0^{-2}$ *for all* $x \in B(p, r_0)$ *and*
2. *(Volume collapsed)* $V(B(p, r_0)) \leq \kappa r_0^d$.

If these are not satisfied, then we say the manifold is κ-noncollapsed at p at the scale r_0. Is this a reasonable definition? Here are some observations:

- By Cheeger's theorem, this would imply a lower bound on injectivity radius.
- Note that if the one is on a ball where the curvature is large, then the manifold is automatically κ-noncollapsed at that scale.
- By the discussion above, every manifold is κ-noncollapsed at a small enough scale and κ smaller than the constant for a Euclidean ball.
- This definition is scale independent in the following sense. If we consider $\bar{g} = r_0^{-2} g$, then the conditions are

$$|\mathrm{Rm}(\bar{g})|_{\bar{g}} \leq 1$$
$$V(B_{\bar{g}}(p, 1)) \leq \kappa.$$

- The sphere $\mathbb{S}^n$ is κ-noncollapsed at scales r_0 less than the diameter (for a suitable choice of κ). The bounded normalized curvature assumption of the definition are satisfied for $r_0 \leq 1$, although one might argue that there is really no local collapsing at scales less than π. Apparently this will not be important for our argument. At large scales, the curvature assumption fails.
- The flat torus has $|\mathrm{Rm}| = 0$, and so it is noncollapsed at scales less than the injectivity radius. The curvature assumption is valid for large scales, but for a given κ, if r_0 is taken large enough, the torus must be collapsed, since the volume is never larger than the volume of the torus.
- We want to consider whether there exists a κ such that a manifold is κ-noncollapsed at large and small scales. Certainly, one can make κ small enough (say, less than the constant for the area of a Euclidean ball) so that a manifold is κ-collapsed at many scales, but this is not of use to us.

Perelman adapted these ideas to Ricci flow (time dependent metrics) as follows.

Definition 50 *Let $\left(M^d, g(t)\right)$ be a solution to Ricci flow and let $\kappa > 0$. Then Ricci flow is κ-collapsed at a point (t_0, x_0) in spacetime at a scale r_0 if:*

1. *(Bounded normalized curvature)*

$$|\mathrm{Rm}(t,x)|_{g(t)} \leq r_0^{-2}$$

   for all $(t,x) \in \left[t_0 - r_0^2, t_0\right] \times B_{g(t_0)}(x_0, r_0)$, and

2. *(Collapsed volume)*

$$V\left(B_{g(t_0)}(x_0, r_0)\right) \leq \kappa r_0^d.$$

Otherwise, we say that the solution is κ-noncollapsed at p at scale r_0.

Remark 51 *Notice that the assumptions require Ricci flow to exist on the time interval $[t_0 - r_0^2, t_0]$.*

Remark 52 *As remarked above, for each κ smaller than the volume of a unit ball in Euclidean space, there is a r_* so that M is κ-noncollapsed at scales less than r_*. For Ricci flow, one can still find r_*, but it will, in general depend on t_0. As t_0 goes to a singular time, it may be possible that $r_* \to 0$. This is what we would like to rule out, as we shall see.*

Remark 53 *Notice that κ is dimensionless.*

Remark 54 *If $g(t)$ is κ-noncollapsed at (t_0, x_0) at the scale r_0, we see that $Kg\left(t_* + \frac{t}{K}\right)$ is κ-noncollapsed at $(K(t_0 - t_*), x_0)$ at the scale of $K^{-1/2} r_0$.*

Here is a typical noncollapsing theorem along the lines of Perelman.

Theorem 55 (Perelman's noncollapsing theorem, first version) *Let $(M, g(t))$ be a solution to the Ricci flow on compact 3-manifolds for $t \in [0, T_0]$ such that at $t = 0$ we have*

$$|\mathrm{Rm}(p)|_{g(0)} \leq 1$$
$$V\left(B_{g(0)}(p,1)\right) \geq \omega$$

for all $p \in M$ and $\omega > 0$ fixed. Then there exists $\kappa = \kappa(\omega, T_0) > 0$ such that the Ricci flow is κ-noncollapsed for all $(t_0, x_0) \in [0, T_0] \times M$ and scales $0 < r_0 < \sqrt{t_0}$.

A big point of this theorem is that it rules out a limit of $\Sigma \times \mathbb{R}$ where Σ is the cigar soliton solution of Hamilton. The manifold $\Sigma \times \mathbb{R}$ is essentially a fixed point of Ricci flow (when we consider it a flow on metric spaces, not Riemannian metrics). Σ is a positive curvature metric on $\mathbb{R}^2$ which has maximum curvature at the origin and is asymptotic to a cylinder as one moves away from the origin.

This implies that $\Sigma \times \mathbb{R}$ has volume $V(B(0,r))$ asymptotic to Cr^2 for large r (For a cylinder, notice that large balls of radius r have volumes asymptotic to Cr, not Cr^2.) Thus, $\Sigma \times \mathbb{R}$ is not κ-noncollapsed at large scales (r_0 large).

Consider the blowups $(M, g_n(t))$ defined above. Note that we have a κ such that the Ricci flow $g(t)$ is κ-noncollapsed for time interval $[0, T_0]$ for any $T_0 < T$. Thus, by Remark 54 we must have that $g_n(t)$ is κ-noncollapsed at the scale of $L_n^{-1} r_0$ for the time interval $\left[-\frac{t_n}{L_n^2}, \frac{T-T_0}{L_n^2}\right)$. As n becomes large, we see that $g_n(t)$ becomes κ-noncollapsed at all scales, which is not true for $\Sigma \times \mathbb{R}$.

We will describe this in more detail later in the course. We will now move to proving this theorem, the subject of the next few lectures.

Chapter 7

Ricci flow from energies

7.1 Gradient flow

Formulating an equation as a gradient flow has many advantages. Consider the heat equation

$$\frac{\partial f}{\partial t} = \triangle f$$

on a compact Riemannian manifold. It is easy to see that if one considers the energy

$$E(f) = \frac{1}{2} \int_M |\nabla f|_g \, dV,$$

that if we take the time derivative of the energy when f satisfies the heat equation, we get

$$\begin{aligned} \frac{dE}{dt}(f) &= \int_M \nabla f \cdot \nabla \left(\frac{\partial}{\partial t} f \right) dV \\ &= - \int_M \triangle f \left(\frac{\partial}{\partial t} f \right) dV \\ &= - \int_M (\triangle f)^2 \, dV \leq 0. \end{aligned}$$

Thus we immediately get that the energy is decreasing and that stationary points are harmonic functions, i.e., functions which satisfy $\triangle f = 0$. This monotonicity also tells us that f cannot have periodic solutions which are not fixed points, for if $f(t_1, x) = f(t_2, x)$ for all x, then $E(f(t_1, \cdot)) = E(f(t_2, \cdot))$, and the monotonicity implies that $\triangle f = 0$ for $t \in [t_1, t_2]$.

The monotonicity is true in general for a gradient flow. If one has an energy $E(f)$, one defined the gradient flow as

$$\frac{\partial f}{\partial t} = \pm grad(E),$$

where the gradient vector $grad\,(E)$ is given so that

$$dE\,(X) = g\,(X, grad\,(E))\,,$$

for some metric g on the space of functions. In our case that g is the L^2 metric (which uses the Riemannian metric g on M).

It would be nice to represent Ricci flow in this way. It is not at all trivial to do this.

7.2 Ricci flow as a gradient flow

An obvious choice of functional is the Einstein-Hilbert functional:

$$EH\,(g) = \int_M R dV.$$

To calculate its variation, recall that if we have a variation of the metric $(\delta g)_{ij} = h_{ij}$, then we get

$$\delta R\,(h) = -\,\langle \mathrm{Rc}, h\rangle - \triangle\,\mathrm{tr}_g\,(h) + \mathrm{div}\,\mathrm{div}\,h.$$

It is not hard to see that since

$$\delta \log\det g = g^{ij} h_{ij}$$

so

$$\begin{aligned}\delta\sqrt{\det g} &= \delta\exp\left(\frac{1}{2}\log\det g\right)\\ &= \frac{1}{2}\left(\mathrm{tr}_g\,h\right)\sqrt{\det g}\end{aligned}$$

so

$$\delta\,(dV) = \frac{1}{2}\left(\mathrm{tr}_g\,h\right)dV.$$

Using the above formula we get

$$\begin{aligned}\delta\,(EH)\,(h) &= \int_M \left(-\,\langle \mathrm{Rc}, h\rangle - \triangle\,\mathrm{tr}_g\,(h) + \mathrm{div}\,\mathrm{div}\,h + \frac{1}{2}R\,(\mathrm{tr}_g\,h)\right)dV\\ &= \int_M \left(-\,\langle \mathrm{Rc}, h\rangle + \frac{1}{2}R\,(\mathrm{tr}_g\,h)\right)dV\\ &= \int_M \left\langle \frac{1}{2}Rg - \mathrm{Rc}, h\right\rangle dV.\end{aligned}$$

Remark 56 *Here we used the divergence theorem for a compact Riemannian manifold, which says that*

$$\int \langle \mathrm{div}\,T, S\rangle\,dV = -\int \langle T, \nabla S\rangle\,dV,$$

for any n-tensor T and $(n-1)$-tensor S. More explicitly,

$$\int g^{i_1 j_1}\cdots g^{i_n j_n} g^{j i_0} \nabla_j T_{i_0 i_1 \cdots i_n} S_{j_1 j_2 \cdots j_n} dV = \int g^{i_1 j_1}\cdots g^{i_n j_n} g^{j i_0} T_{i_0 i_1 \cdots i_n} \nabla_j S_{j_1 j_2 \cdots j_n} dV.$$

So critical points of the Einstein-Hilbert functional satisfy the Einstein equation. The gradient flow would be

$$\frac{\partial}{\partial t} g = -2\left(\mathrm{Rc} - \frac{1}{2} R g\right).$$

The problem is that this flow is not parabolic and there is no existence theory for such equations.

Let's try a new tactic. Replace dV by a fixed measure dm. Then the functional

$$H(g) = \int_M R\,dm$$

satisfies the variation

$$\delta H(h) = \int_M \left(-\langle \mathrm{Rc}, h\rangle - \triangle \operatorname{tr}_g(h) + \operatorname{div}\operatorname{div} h\right) dm.$$

You don't lose the last two terms with the divergence theorem, since that only works with the volume measure. However, we can consider the Radon-Nikodym derivative and write

$$dm = \frac{dm}{dV} dV$$

for a positive function $\frac{dm}{dV}$. We can write

$$\frac{dm}{dV} = e^{-f}.$$

Then,

$$\delta H(h) = \int_M \left(-\langle \mathrm{Rc}, h\rangle - \triangle \operatorname{tr}_g(h) + \operatorname{div}\operatorname{div} h\right) e^{-f} dV$$

can be integrated by parts to get

$$\begin{aligned}
\delta H(h) &= \int_M \left(-\langle \mathrm{Rc}, h\rangle - \langle \nabla \operatorname{tr}_g(h), \nabla f\rangle + \operatorname{div} h \cdot \nabla f\right) e^{-f} dV \\
&= \int_M \left(-\langle \mathrm{Rc}, h\rangle + \operatorname{tr}_g(h) \triangle f - \operatorname{tr}_g(h) |\nabla f|^2 - \left\langle h, \nabla^2 f\right\rangle + h(\nabla f, \nabla f)\right) e^{-f} dV \\
&= \int_M \left\langle -\mathrm{Rc} - \nabla^2 f + \left(\triangle f - |\nabla f|^2\right) g + \nabla f \nabla f, h\right\rangle e^{-f} dV.
\end{aligned}$$

Remark 57 *Tao often uses $\langle\cdot,\cdot\rangle$ to denote the Euclidean metric locally, and so explicitly puts in g^{ij}'s in this case. We will understand that $\langle\cdot,\cdot\rangle$ requires the metric, and so when quantities like this are differentiated, we also need to differentiate the metric, as we will see below.*

We need to add another term, and so we get

$$F(g) = \int_M \left(R + |\nabla f|^2\right) e^{-f} dV = \int_M \left(R + \left|\nabla \log \frac{dm}{dV}\right|^2\right) dm.$$

We now need that

$$\begin{aligned} 0 = \delta(dm) = \delta\left(e^{-f} dV\right) &= -(\delta f) e^{-f} dV + \frac{1}{2} \operatorname{tr}_g h e^{-f} dV \\ &= \left(-\delta f + \frac{1}{2} \operatorname{tr}_g h\right) e^{-f} dV \end{aligned}$$

Thus we have that

$$\delta f = \frac{1}{2} \operatorname{tr}_g h$$

Let

$$E(g) = \int \left(g^{ij} \nabla_i f \nabla_j f\right) e^{-f} dV$$

where $dm = e^{-f} dV$ is fixed (so that $f = -\log dm/dV$ and δf is expressed as above). We can then compute

$$\begin{aligned} \delta E(h) &= \delta\left[\int \left(g^{ij} \nabla_i f \nabla_j f\right) e^{-f} dV\right] \\ &= \int \left(-\langle h, \nabla f \nabla f\rangle + 2\langle \nabla f, \nabla(\delta f)\rangle - |\nabla f|^2 \delta f + \frac{1}{2}|\nabla f|^2 \operatorname{tr}_g h\right) e^{-f} dV \\ &= \int \left(-\langle h, \nabla f \nabla f\rangle - 2\triangle f(\delta f) + |\nabla f|^2 \delta f + \frac{1}{2}|\nabla f|^2 \operatorname{tr}_g h\right) e^{-f} dV \\ &= \int \left(\left\langle h, -\nabla f \nabla f - (\triangle f) g + |\nabla f|^2 g\right\rangle\right) e^{-f} dV. \end{aligned}$$

Now, since $F = H + E$, we have

$$\delta F(h) = \int_M \left\langle -\operatorname{Rc} - \nabla^2 f, h\right\rangle e^{-f} dV.$$

Thus the gradient flow of $-2F$ is

$$\frac{\partial g}{\partial t} = -2\operatorname{Rc}(g) - 2\nabla^2 f.$$

This is almost Ricci flow, but not quite. The f is changing, too, by the equation

$$\frac{\partial f}{\partial t} = -\triangle f - R.$$

This is a backward heat equation, which it turns out will make it useful to probe backwards.

Notice that

$$2\nabla^2 f = \mathcal{L}_{\nabla f} g,$$

the Lie derivative of g in the direction ∇f. This means that the flow above differs from Ricci flow by a diffeomorphism. Instead, we can consider $\bar{g} = \phi^* g$, where ϕ is the flow of diffeomorphisms generated by ∇f, and we will see that $\bar{g}$ evolves by Ricci flow. Furthermore, f will differ by a Lie derivative, and

$$\mathcal{L}_{\nabla f} f = df\,(\nabla f) = |\nabla f|^2$$

(where is the metric in here? It is in ∇f, which is a vector field gotten by raising the index on df) and so under the new flow, $\bar{f} = f \circ \phi$ evolves by

$$\frac{\partial \bar{f}}{\partial t} = -\triangle_{\bar{g}} \bar{f} - \bar{R} + \left|\nabla \bar{f}\right|^2_{\bar{g}}.$$

Example 58 (Fundamental, important example) *Let (M, g) be Euclidean space $M = \mathbb{R}^d$ and let*

$$f\,(t, x) = \frac{|x|^2}{4\tau} + \frac{d}{2} \log 4\pi\tau = -\log\left[(4\pi\tau)^{-d/2} e^{-|x|^2/(4\tau)}\right]$$

where $\tau = t_0 - t$, Notice that $e^{-f} dx$ is the Gaussian measure, which solves the backward heat equation (the fundamental solution to the heat equation). If $t < t_0$, this choice of g and f satisfy the equations

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Rc}\,(g) \tag{7.1}$$

$$\frac{\partial f}{\partial t} = -\triangle f - R + |\nabla f|^2. \tag{7.2}$$

We can check:

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial}{\partial t}\left[\frac{|x|^2}{4\tau} + \frac{d}{2}\log 4\pi\tau\right] \\ &= \frac{|x|^2}{4\tau^2} - \frac{d}{2\tau}.\end{aligned}$$

$$\begin{aligned}\nabla f &= \frac{x}{2\tau} \\ |\nabla f|^2 &= \frac{|x|^2}{4\tau^2} \\ \triangle f &= \frac{d}{2\tau}\end{aligned}$$

so it works.

Notice that when we pull back by ϕ, the measure $\phi^* dm$ is not static. Thus it makes sense to rewrite the functional as

$$F\,(M, g, f) = \int_M \left(R + |\nabla f|^2\right) e^{-f} dV.$$

Notice that this functional is invariant under diffeomorphism, i.e.,

$$F(M, \phi^* g, f \circ \phi) = F(M, g, \phi)$$

for any diffeomorphisms ϕ. We also have that under the coupled flows (7.1) and (7.2),

$$\frac{\partial F}{\partial t}(M, g, f) = 2\int \left|\mathrm{Rc} + \nabla^2 f\right|^2 e^{-f} dV.$$

Thus we have that F is monotone increasing under Ricci flow. Unfortunately, now we have to explicitly deal with this quantity f. To eliminate this, we take the infimum:

$$\lambda(M, g) = \inf\left\{F(M, g, f) : \int_M e^{-f} dV = 1\right\}$$

(the infimum is over f). Using the following exercise, we can show that λ is finite.

Exercise 59 *Show that $\lambda(M, g)$ is the smallest number for which one has the inequality*

$$\int_M \left(4|\nabla u|_g^2 + Ru^2\right) dV \geq \lambda \int_M u^2 dV,$$

where u is in $H^1(M) = W^{1,2}(M)$, the Sobolev space of functions with 1 derivative in L^2 (so it has norm $\|f\|_{H^1} = \int \left(|\nabla f|_g^2 + f^2\right) dV$ for C^1 functions). Hint: show that we can assume u is positive and then write $u = e^{-f/2}$. Thus λ is the smallest eigenvalue of the operator $-4\triangle_g u + R$.

Using the exercise, one sees that the fact that every compact manifold satisfies a Poincaré inequality,

$$\int_M |\nabla u|^2 dV \geq c(d) \int_M u^2 dV,$$

implies that λ is bounded below, basically, by the best constant in the Poincaré inequality plus $\min R$. Note that the Poincaré inequality constant depends on the dimension. We will see later a similar inequality for which the constant does not depend on dimension.

Furthermore, one can prove that λ is realized by a positive function $u = e^{-f/2}$ with $\|u\|_{L^2(M)} = 1$. Note that $H^1(M)$ embeds compactly into $L^2(M)$ since M is compact. Thus if we take a minimizing sequence $\{u_n\}$ in H^1, there is a subsequence (which we also denote by $\{u_n\}$ which converges in L^2 to a function u. Now consider:

$$\int \left(|\nabla u_n|^2 + Ru_n^2\right) dV + \int \left(|\nabla u_m|^2 + Ru_m^2\right) dV$$
$$= \frac{1}{2}\int \left(|\nabla(u_n - u_m)|^2 + R(u_n - u_m)^2\right) dV + \frac{1}{2}\int \left(|\nabla(u_n + u_m)|^2 + R(u_n + u_m)^2\right) dV$$

so

$$\frac{1}{2}\int\left(\left|\nabla\left(u_n-u_m\right)\right|^2+R\left(u_n-u_m\right)^2\right)dV=\int\left(\left|\nabla u_n\right|^2+Ru_n^2\right)dV+\int\left(\left|\nabla u_m\right|^2+Ru_m^2\right)dV$$
$$-\frac{1}{2}\int\left(\left|\nabla\left(u_n+u_m\right)\right|^2+R\left(u_n+u_m\right)^2\right)dV.$$

The right side goes to zero in the limit since the terms go to λ, λ and -2λ. Also, we know that

$$-\min|R|\left\|u_n-u_m\right\|_{L_2}\leq\int R\left(u_n-u_m\right)^2dV\leq\min|R|\left\|u_n-u_m\right\|_{L_2}$$

and each term goes to zero. Thus we know that the sequence $\{u_n\}$ is Cauchy in H^1 and since H^1 is complete, it must converge to a function in H^1.

Since λ is attained at a function, we can sometimes prove inequalities like the following (taken from the notes of Kleiner and Lott):

Let $h(s,t,x)$ be a two-parameter family of functions such that

$$h(s,t_0)=f_*(t_0+s)$$
$$\frac{\partial h}{\partial t}=-\triangle h+-R+|\nabla h|^2.$$

There is a solution to this for $t\leq t_0$ since the equation is backwards elliptic. (Note: we have only shown that f_* is in H^1, so we mean a weak solution to the parabolic equation. We could also show that f_*, as a minimizer, satisfies a particular elliptic equation, which implies that f_* is smooth by elliptic regularity theory.)

$$\begin{aligned}\lambda(t_0)&\leq F(M,g(t_0),h(s,t_0-s))\\&=F(M,g(t_0+s),h(s,t_0))-2\int_0^s\left[\int_M\left|\operatorname{Rc}(g(t_0+\sigma))+\nabla^2h(s,t_0+\sigma)\right|^2e^{-h(s,t_0+\sigma)}dV\right]d\sigma.\end{aligned}$$

We then have

$$\lambda(t_0)\leq\lambda(t_0+s)-2\int_0^s\left[\int_M\left|\operatorname{Rc}(g(t_0+\sigma))+\nabla^2h(s,t_0+\sigma)\right|^2e^{-h(s,t_0+\sigma)}dV\right]d\sigma$$

and so

$$\begin{aligned}\frac{\partial\lambda}{\partial t}(t_0)&=\lim_{s\to0+}\frac{\lambda(t_0+s)-\lambda(t_0)}{s}\\&\geq\lim_{s\to0+}\frac{2}{s}\int_0^s\left[\int_M\left|\operatorname{Rc}(g(t_0+\sigma))+\nabla^2h(s,t_0+\sigma)\right|^2e^{-h(s,t_0+\sigma)}dV\right]d\sigma\\&=2\int_M\left|\operatorname{Rc}(g(t_0))+\nabla^2h(0,t_0)\right|^2e^{-h(0,t_0)}dV\\&=2\int_M\left|\operatorname{Rc}(g(t_0))+\nabla^2f_*(t_0)\right|^2e^{-f_*}dV\end{aligned}$$

We can now derive

$$\begin{aligned}
\frac{\partial \lambda}{\partial t} &\geq 2 \int \left| \operatorname{Rc} + \nabla^2 f_* \right|^2 e^{-f_*} dV \\
&\geq \frac{2}{3} \int \left(R + \triangle f_* \right)^2 e^{-f_*} dV \\
&\geq \frac{2}{3} \left[\int \left(R + \triangle f_* \right) e^{-f_*} dV \right]^2 \\
&= \frac{2}{3} \left[\int \left(R + \left| \nabla f_* \right|^2 \right) e^{-f_*} dV \right]^2 \\
&= \frac{2}{3} \lambda^2 .
\end{aligned}$$

7.3 Perelman entropy

We now wish to make our functional scale invariant (so that we get a critical quantity, not just subcritical). In particular, we know that

$$\frac{dF_m}{dt} \left(M, g \right) = 2 \int \left| \operatorname{Rc} + \nabla^2 f \right|^2 e^{-f} dV$$

is fixed (under the gradient flow) if

$$\operatorname{Rc} = -\nabla^2 f,$$

i.e., if (M, g) is a gradient Ricci soliton. We wish to have a new functional which is fixed if (M, g) is a gradient shrinking soliton, i.e.,

$$\operatorname{Rc} = -\nabla^2 f + \frac{1}{2\tau} g$$

for some $\tau > 0$. A round sphere is a gradient shrinking soliton, so it makes sense that we would want something like this. Under the Ricci flow, this structure is preserved except that τ decreases at a constant rate.

First note that if we consider the Nash entropy

$$N_m \left(M, g \right) = \int_M \frac{dm}{dV} \log \frac{dm}{dV} dV = \int_M \left(\log \frac{dm}{dV} \right) dm = - \int f dm$$

then

$$\begin{aligned}
\frac{dN_m}{dt} &= - \int_M \left(-\triangle f - R \right) dm \\
&= \int_M \left(R + \left| \nabla f \right|^2 \right) dm = F_m \left(M, g \right) .
\end{aligned}$$

This will come in handy. Now, suppose we want a quantity $W(M, g)$ such that

$$\frac{dW}{dt} = \int_M \left| \operatorname{Rc} + \nabla^2 f - \frac{1}{2\tau} g \right|^2 dm.$$

But in this case, would not have scale invariance for W, since t scales like distance squared, so the integrand should scale like distance squared. We will fix this by assuming

$$\frac{d\tau}{dt} = -1$$

and trying for

$$\frac{dW}{dt} = 2\tau \int_M \left| \operatorname{Rc} + \nabla^2 f - \frac{1}{2\tau} g \right|^2 dm. \tag{7.3}$$

Now, to find such a quantity, consider

$$\left| \operatorname{Rc} + \nabla^2 f - \frac{1}{2\tau} g \right|^2 = \left| \operatorname{Rc} + \nabla^2 f \right|^2 - \frac{1}{\tau}(R + \triangle f) + \frac{d}{4\tau^2}.$$

Thus we have that

$$\begin{aligned} 2\tau \int_M \left| \operatorname{Rc} + \nabla^2 f - \frac{1}{2\tau} g \right|^2 dm &= \tau \frac{dF_m}{dt} - 2F_m + \frac{d}{2\tau} \\ &= \frac{d}{dt}\left(\tau F_m - N_m - \frac{d}{2}\log\tau \right). \end{aligned}$$

This is what our W_m would be. However, as we did last time, we wanted to reparametrize so that

$$dm = e^{-f} dV$$

where dV is evolving according to Ricci flow evolution. This time, we will change

$$\tilde{f} = f - \frac{d}{2}\log(4\pi\tau)$$

so that

$$dm = e^{-f} dV = (4\pi\tau)^{-d/2} e^{-\tilde{f}} dV.$$

Remark 60 *This looks like the heat kernel for Euclidean space, which is why this particular normalization is given.*

Note that the preservation of dm implies that

$$\frac{d}{2\tau} - \frac{\partial \tilde{f}}{\partial t} + \frac{1}{2}\operatorname{tr} h = 0.$$

Under the gradient flow

$$\frac{\partial g}{\partial t} = -2\operatorname{Rc} - 2\nabla^2 f,$$

we have

$$\frac{\partial \tilde{f}}{\partial t} = \frac{d}{2\tau} - R - \triangle f.$$

Thus we get for

$$\begin{aligned} W_m(M, g, \tau) &= \tau F_m - N_m - \frac{d}{2} \log \tau \\ &= \int \left[\tau \left(R + \left| \nabla \tilde{f} \right|^2 \right) + \tilde{f} - \frac{d}{2} \log (4\pi) \right] (4\pi\tau)^{-d/2} e^{-\tilde{f}} dV \end{aligned}$$

Actually, we usually renormalize this to vanish in the Euclidean case, and so we can change the d term appropriately to

$$W_m(M, g, \tau) = \int \left[\tau \left(R + \left| \nabla \tilde{f} \right|^2 \right) + \tilde{f} - d \right] (4\pi\tau)^{-d/2} e^{-\tilde{f}} dV.$$

We can also define the Perelman entropy as the functional

$$W(M, g, f, \tau) = \int \left[\tau \left(R + |\nabla f|^2 \right) + f - d \right] (4\pi\tau)^{-d/2} e^{-f} dV$$

where g is a Riemannian metric, f is a function on M, and τ is a positive constant.

If g satisfies the Ricci flow, then we need to pull back f to get the three evolutions

$$\frac{\partial g}{\partial t} = -2 \operatorname{Rc} \tag{7.4}$$

$$\frac{\partial f}{\partial t} = \frac{d}{2\tau} - R - \triangle f + |\nabla f|^2 \tag{7.5}$$

$$\frac{d\tau}{dt} = -1.$$

Under these three, the Perelman Entropy satisfies

$$\frac{dW}{dt}(M, g, f, \tau) = 2\tau \int_M \left| \operatorname{Rc} + \nabla^2 f - \frac{1}{2\tau} g \right|^2 (4\pi\tau)^{-d/2} e^{-f} dV.$$

We would like to find a minimum over all functions f and constants τ so that we have an invariant of the Riemannian manifold. However, it is not yet clear that such an infimum exists. Recall that last time, the existence followed from a Poincaré inequality. In this scale invariant setting, the existence of a minimizer will follow from a log-Sobolev inequality.

7.4 Log-Sobolev inequalities

Let's first consider what happens if g is the Euclidean metric. We would like to switch to a function which looks like the heat kernel, namely,

$$u = (4\pi\tau)^{-d/2} e^{-f}.$$

Recall that our model case is when $f = |x|^2 / (4\tau)$, in which case this is precisely the backwards heat kernel. The backwards heat kernel satisfies:

$$\frac{\partial u}{\partial t} = -\triangle u$$

for $\tau > 0$ and

$$\lim_{\tau \to 0^-} u(\tau, x) = \delta_0(x),$$

weakly, where δ_0 is the delta function. Furthermore, one can check that

$$\int_{\mathbb{R}^d} u dx = \int_{\mathbb{R}^d} (4\pi\tau)^{-d/2} e^{-|x|^2/(4\tau)} dx = 1 \tag{7.6}$$

for any τ. The backwards heat kernel can be used to solve the heat equation with some given final conditions, e.g., to solve

$$\begin{aligned} \frac{du}{dt} &= -\triangle u \\ u(T, x) &= f(x), \end{aligned}$$

we see that the convolution

$$u(t, x) = \int_{\mathbb{R}^d} f(y) (4\pi\tau)^{-d/2} e^{-|x-y|^2/(4\tau)} dy$$

is a solution.

Exercise 61 *Show that all of this is true. Hint: to show (7.6), turn the integral into polar coordinates and assume the dimension is at least* 2*. For the dimension 1 case, there is a trick involving turning it into a dimension 2 integral and separating.*

We can check that for g Euclidean and f as above, we have

$$W(M, g, f, \tau) = \int \left[\frac{|x|^2}{2\tau} - d \right] (4\pi\tau)^{-d/2} e^{-|x|^2/(4\tau)} dx.$$

One can show that this is zero since

$$W(M, g, f, \tau) = \int \left[2\tau \left(|\nabla f|^2 - \triangle f \right) \right] (4\pi\tau)^{-d/2} e^{-f} dx,$$

and integrating by parts (needs to be justified) shows this is equal to zero.

Now we re-write W by replacing f with u. We see that (remembering still we are in Euclidean space),

$$W = \int \left[\tau \frac{|\nabla u|^2}{u^2} - u \log u \right] dx - \frac{d}{2} \log(4\pi\tau) - d$$

using identities such as

$$
\begin{aligned}
u &= (4\pi\tau)^{-d/2} e^{-f}. \\
\log u &= -\frac{d}{2}\log(4\pi\tau) - f \\
|\nabla u|^2 &= (4\pi\tau)^{-d} |\nabla f|^2 e^{-2f} \\
|\nabla f|^2 &= \frac{|\nabla u|^2}{u^2}.
\end{aligned}
$$

Tao shows that one can show that $W \geq 0$, which implies a log-Sobolev inequality

$$\tau \int \frac{|\nabla u|^2}{u^2} dx \geq \int u \log u dx + \frac{d}{2}\log(4\pi\tau) + d,$$

or as it is usually stated, with $\phi^2 = u$,

$$4\tau \int |\nabla \phi|^2 dx \geq \frac{1}{\tau} \int \phi^2 \log \phi^2 dx + \frac{d}{2}\log(4\pi\tau) + d.$$

For the general case, we have

$$W(M, g, f, \tau) = \int \left[\tau \left(Ru + \frac{|\nabla u|^2}{u^2}\right) - u \log u\right] dV - \frac{d}{2}\log(4\pi\tau) - d$$

One can show that the

$$W(M, g, f, \tau) \geq -C(M, g, \tau).$$

This implies essentially a log-Sobolev inequality, i.e.,

$$\tau \int R\phi^2 dV + \tau \int 4|\nabla\phi|^2 dV \geq -C + \int \phi^2 \log \phi^2 dV + \frac{d}{2}\log(4\pi\tau) + d.$$

In fact, we can take the

$$\mu(M, g, \tau) = \inf\left\{W(M, g, f, \tau) : \int (4\pi\tau)^{-d/2} e^{-f} dV = 1\right\},$$

which is the best possible constant $-C$. It can be shown that μ is finite, which is what we could call a log-Sobolev inequality.

We can now show that if $g(t)$ is a solution to Ricci flow on $t \in [0, T_0]$ and $\tau = T_0 - t$, then $\mu(M, g, \tau)$ is increasing. The first exercise is important:

Exercise 62 *Show that $\mu(M, g, \tau) = W(M, g, f_*, \tau)$ for a function $f_* \in H^1(M)$. (Not quite true... What is the true statement? Hint: you need to change to a new function ϕ.)*

Once we know that μ is realized by a function, we can show that $\mu(M, g(t), \tau(t))$ is increasing as follows. Calculate $\mu(M, g(t_0), \tau(t_0)) = W(M, g(t_0), f_*(t_0), \tau(t_0))$ for some minimizer $f_*(t_0)$. For any time $t \leq t_0$, we can solve the equation for f in 7.4 backwards to t with initial condition $f(t_0, x) = f_*(t_0, x)$ (since f_* is in $H^1(M)$, there exists a weak solution to this parabolic flow). We know that

$$\begin{aligned}\mu(M, g(t), \tau(t)) &\leq W(M, g(t), f(t), \tau(t)) \\ &\leq W(M, g(t_0), f_*(t_0), \tau(t_0)) = \mu(M, g(t_0), \tau(t_0)).\end{aligned}$$

7.5 Noncollapsing

We will now show that log-Sobolev inequalities imply noncollapsing. Suppose we have a ball $B(p, \sqrt{\tau})$ with bounded normalized curvature, i.e.,

$$|\mathrm{Rm}(x)| \leq \frac{1}{\tau}$$

for $x \in B(p, \sqrt{\tau})$. Then $|R|\tau \leq c(d)$ for some constant depending only on dimension. Then the log-Sobolev inequality can be rewritten as

$$c(d)\int \phi^2 dV + \tau \int 4|\nabla\phi|^2 dV \geq \mu(M, g, \tau) + \int \phi^2 \log \phi^2 dV + \frac{d}{2}\log(4\pi\tau) + d.$$

Suppose ϕ is a function supported on $B(p, \sqrt{\tau})$ such that $\int_M \phi^2 dV = 1$. Then Jensen's inequality implies that

$$\begin{aligned}\frac{1}{V(B)}\int_B \phi^2 \log\phi^2 dV &\geq \left(\frac{1}{V(B)}\int_B \phi^2 dV\right)\log\left(\frac{1}{V(B)}\int_B \phi^2 dV\right) \\ &= \frac{1}{V(B)}\log\frac{1}{V(B)},\end{aligned}$$

where $B = B(p, \sqrt{\tau})$. (Recall that Jensen's inequality requires a probability measure.) So

$$\int_M \phi^2 \log\phi^2 dV \geq \log\frac{1}{V(B)}.$$

We now get, for this particular choice of ϕ,

$$4\tau\int |\nabla\phi|^2 dV \geq \mu(M, g, \tau) + \log\frac{\tau^{d/2}}{V(B)} - c'(d).$$

Now we will specialize ϕ even more. Suppose

$$\phi(x) = c\psi\left(\frac{d(x,p)}{\sqrt{\tau}}\right)$$

for some bump function ψ on the real line which is 1 on $[0, 1/2]$ and supported on $[0, 1]$ (technically, we only need half the bump function, which is how I described it). Thus $\phi(x) = c$ on $B(p, \sqrt{\tau}/2)$ and c is such that

$$\int_B \phi^2 dV = 1,$$

so $c \leq V\left(B\left(p, \sqrt{\tau}/2\right)\right)^{-1/2}$. We can choose ϕ so that $|\nabla\phi| \leq c''c/\sqrt{\tau}$ on the ball (for some constant c''), and so

$$4c''\frac{V\left(B\right)}{V\left(B_{1/2}\right)} \geq \mu\left(M, g, \tau\right) + \log\frac{\tau^{d/2}}{V\left(B\right)} - c'\left(d\right).$$

Finally, we can use a Bishop-Gromov volume comparison theorem:

Theorem 63 (Bishop-Gromov comparison) *If $\left(M^d, g\right)$ is a complete Riemannian manifold with*

$$\mathrm{Rc} \geq \left(n-1\right)Kg$$

for some $K \in \mathbb{R}$, then for any $p \in M$, the volume ratio

$$\frac{V\left(B\left(p,r\right)\right)}{V_K\left(B\left(p_K,r\right)\right)}$$

is non-increasing as a function of r, where p_K is a point in the d-dimensional simply connected space of constant sectional curvature K, and $V_K\left(B\left(p_K,r\right)\right)$ is the volume of a ball of radius r in that space.

In particular, we have that

$$\frac{V\left(B\right)}{V_{-1/\tau}\left(B\left(p_{-1/\tau}, \sqrt{\tau}\right)\right)} \leq \frac{V\left(B_{1/2}\right)}{V_{-1/\tau}\left(B\left(p_{-1/\tau}, \sqrt{\tau}/2\right)\right)},$$

and thus there is a $\alpha = \alpha\left(\tau, d\right)$ such that

$$\frac{V\left(B\right)}{V\left(B_{1/2}\right)} \leq \alpha.$$

In fact, α is independent of τ since

$$V_{-1/\tau}\left(B\left(p_{-1/\tau}, \sqrt{\tau}\right)\right) = V_{-1}\left(B\left(p_{-1}, 1\right)\right)$$
$$V_{-1/\tau}\left(B\left(p_{-1/\tau}, \sqrt{\tau}/2\right)\right) = V_{-1}\left(B\left(p_{-1}, 1/2\right)\right).$$

Thus there is a constant c''' which depends on d such that

$$c''' - \mu\left(M, g, \tau\right) \geq \log\frac{\tau^{d/2}}{V\left(B\right)},$$

i.e.,

$$V\left(B\right) \geq \left(e^{\mu - c'''}\right)\tau^{d/2},$$

which implies κ-noncollapsing at a scale $\sqrt{\tau}$ for $\kappa = \exp\left(\mu - c'''\right)$. Let's formulate this into a proposition:

Proposition 64 *There is a constant $c = c\left(d\right)$ depending only on dimension such that if $\mu\left(M^d, g, \tau\right)$ is finite, then for $\kappa = \exp\left(\mu\left(M, g, \tau\right) - c\right)$, the Riemannian manifold $\left(M, g\right)$ is κ-noncollapsed at the scale of $\sqrt{\tau}$.*

Let's collect the facts about μ.

Proposition 65 *The following are true about μ:*

1. *$\mu(M,g,\tau) > -\infty$ for any fixed manifold (M,g) and $\tau > 0$.*
2. *If $(M,g(t))$ satisfies the Ricci flow for $t \in [0,T_0]$ and $\tau(t) = T_0 - t$, then $\mu(M,g(t),\tau(t))$ is increasing.*
3. *There is a constant $c = c(d)$ depending only on dimension such that the Riemannian manifold (M,g) is κ-noncollapsed at the scale of $\sqrt{\tau}$ at every point for $\kappa = \exp(\mu(M,g,\tau) - c)$.*

We can now prove:

Theorem 66 (Perelman's noncollapsing theorem, first version) *Let $(M,g(t))$ be a solution to the Ricci flow on compact 3-manifolds for $t \in [0,T)$ such that at $t = 0$ we have*

$$|\mathrm{Rm}(p)|_{g(0)} \leq 1$$
$$V\left(B_{g(0)}(p,1)\right) \geq \omega$$

for all $p \in M$ and $\omega > 0$ fixed. For any $\rho > 0$, there exists $\kappa = \kappa(\omega,T,\rho) > 0$ such that the Ricci flow is κ-noncollapsed for all $(t_0,x_0) \in [0,T) \times M$ and scales $0 < r_0 < \rho$. We could also take $\rho = \rho(t)$ and get a similar result, as long as $\rho(t)$ is uniformly bounded on $[0,T)$.

Proof. We already showed that for a given τ and metric, $\mu(M,g,\tau)$ has a lower bound. For any r_0^2, we see by monotonicity that

$$\mu\left(M,g(t),r_0^2\right) \geq \mu\left(M,g(0),r_0^2+t\right).$$

Thus we have that if

$$\mu_0 = \inf\left\{\mu\left(M,g(0),r^2\right) : r^2 \in (0,\rho+T)\right\}$$

then

$$\mu\left(M,g(t),r_0^2\right) \geq \mu_0.$$

Thus $(M,g(t))$ is κ-noncollapsed at the scale of r_0 for all

$$\kappa = \exp(\mu_0 - c) \leq \exp\left[\mu\left(M,g(t),r_0^2\right) - c\right].$$

We need to see that μ_0 is not $-\infty$. Since T is finite, there is no problem at the top of the interval for r^2. It can be shown that as $r^2 \to 0^+$, $\mu\left(M,g(0),r^2\right) \to 0$ (in the interest of time, we will not show this) and so there is no problem at the other side. ■

Remark 67 *This is a bit stronger than what I proposed in an earlier lecture. I think Tao was thinking about future incarnations of this theorem, which is why he formulated as he did.*

Chapter 8

Ricci flow for attacking geometrization

8.1 Introduction

I would like to go back to the general program and see what we need to learn about. Much of this if from Morgan-Tian, with help from other sources.

Recall that we wish to perform surgery when the Ricci flow comes to a singularity. We will consider a Ricci flow with surgery $(M, g(t))$ defined for $0 \leq t < T < \infty$ which satisfies the following properties:

1. (Normalized initial conditions) We have

   $$|\mathrm{Rm}(g(0))| \leq 1$$

   and

   $$V(B(x,1)) \geq \frac{1}{2} V\left(B_{\mathbb{E}^3}(0,1)\right)$$

   for any $x \in M$.

2. (Curvature pinching) The curvature is pinched towards positive. This means that as the scalar curvature $R \to +\infty$, the ratio of the absolute value of the smallest eigenvalue of the Riemannian curvature tensor to the largest positive eigenvalue goes to zero.

3. (Noncollapsed) There is a $\kappa > 0$ so that the Ricci flow is κ-noncollapsed.

4. (Canonical neighborhood) Any point with large curvature has a canonical neighborhood.

The key is to show that these conditions both allow surgery and then persist after the surgery. In order to do this, we will need to be more precise with 3 and especially 4.

8.2 Finding canonical neighborhoods

The main way to find these is to take blow-ups as one goes to a singularity. If one takes a sequence $t_i \to T$, where T is a singular time, then the blow ups

$$g_i(t) = M_i g\left(t_i + \frac{t}{M_i}\right).$$

If M_i is comparable to $\sup |\mathrm{Rm}(g(t_i))|$, then $M_i \to \infty$. Furthermore, since $T < \infty$, we have that $g_i(t)$ is defined on the interval $[-M_i t_i, (T - t_i) M_i)$. Thus the limit will definitely be defined on $(-\infty, 0]$, and is thus ancient. Moreover, if we show that g is κ-noncollapsed at some scale r_0, then $g_i(t)$ will be κ-noncollapsed at a scale $r_0\sqrt{M_i}$, and so the limit is κ-noncollapsed on all scales. Finally, one can show that any 3-dimensional ancient solution has nonnegative curvature.

Definition 68 *A κ-solution of Ricci flow is a solution defined for $t \in (-\infty, 0]$ which is κ-noncollapsed on all scales and has nonnegative curvature.*

These are the limits as you go to a finite time singularity. The key is that Perelman was able to classify these solutions in the following way:

1. κ-solutions look like gradient shrinking solitons as $t \to -\infty$.
2. Gradient shrinking solitons in dimension 3 must have finite covers isometric to (i) 3-spheres or (ii) 2-spheres cross $\mathbb{R}$.
3. κ-solutions have canonical neighborhoods.

Now the idea is that as one goes to a singularity, the manifold is like a κ-solution, and thus has canonical neighborhoods.

8.3 Canonical neighborhoods

What is a canonical neighborhood? This is where the surgery should be done, so it needs to be classified sufficiently to allow us to do a careful surgery. It turns out that there are 4 types of canonical neighborhoods:

Definition 69 *A point $x \in M$ is in a (C, ε)-canonical neighborhood if one of the following holds:*

1. *x is contained in a C-component.*
2. *x is contained in an open set which is within ε of round in the $C^{\lfloor 1/\varepsilon \rfloor}$-topology.*
3. *x is contained in the core of a (C, ε)-cap.*
4. *x is in the center of a strong neck.*

Definition 70 *Define the $C^k(X, g_0)$ "norm" on (X, g) to be*

$$\|(X,g)\|^2_{C^k(X,g_0)} = \sup_{x\in X}\left\{|g(x) - g_0(x)|^2_{g_0} + \sum_{\ell=1}^{k}\left|\nabla^{\ell}_{g_0} g(x)\right|^2_{g_0}\right\}.$$

Remark 71 *Technically, the norm should be*

$$|||(X,g)|||^2_{C^k(X,g_0)} = \sup_{x\in X}\left\{|g(x)|^2_{g_0} + \sum_{\ell=1}^{k}\left|\nabla^{\ell}_{g_0} g(x)\right|^2_{g_0}\right\}.$$

The function $\|\cdot\|$ defined above is essentially $\|(X,g)\| = |||(X, g - g_0)|||$. With our current definition, $\|(X,g)\|_{C^k(X,g_0)} = 0$ if $g = g_0$.

Problem 72 *Here is something to think about. Fix $X \subset \mathbb{R}^n$, say bounded. Given a sequence of Riemannian metrics $\{g_i\}$ on X, under what conditions does there exist a subsequence such that (M, g_i) converge to some limit (X, g_∞) for some Riemannian metric g_∞ (i.e., $\|(X,g_i)\|_{C^k(X,g_\infty)} \to 0$ as $i \to \infty$.*

Problem 73 *How could one use this norm idea to compare (X, g) and (X', g') where $X \neq X'$?*

Definition 74 *Let (N, g) be a Riemannian manifold and $x \in N$ a point. Then an ε-neck structure on (N, g) centered at x consists of a diffeomorphism*

$$\phi : S^2 \times \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right) \to N,$$

with $x \in \phi\left(S^2 \times \{0\}\right)$, such that

$$\|(N, R(x)\,\phi^* g)\| < \varepsilon$$

where the norm is with respect to $C^{\lfloor 1/\varepsilon \rfloor}\left(S^2 \times \left(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right), g_{std}\right)$, where g_{std} is the product of the metric with curvature $1/2$ on S^2 with the Euclidean metric on the interval. We say N is a ε-neck centered at x.

Definition 75 *A compact, connected, Riemannian manifold (M, g) is called a C-component if*

1. *M is diffeomorphic to S^3 or $\mathbb{RP}^3$,*
2. *(M, g) has positive sectional curvature,*
3. *For every 2-plane P in TX,*

$$\frac{1}{C} < \frac{\inf_P K(P)}{\sup_{y\in M} R(y)},$$

4.
$$C^{-1}\sup_{y\in M}\frac{1}{\sqrt{R(y)}}<\operatorname{diam}(M)<C\inf_{y\in M}\frac{1}{\sqrt{R(y)}}.$$

Definition 76 *A compact, connected 3-manifold (M,g) is within ε of round in the $C^{\lfloor 1/\varepsilon\rfloor}$-topology if there exists a constant $\rho>0$, a compact manifold (Z,g_0) of constant curvature $+1$, and a diffeomorphism*

$$\phi : Z\to M$$

such that
$$\|(Z,\phi^*(\rho g))\|_{C^{\lfloor 1/\varepsilon\rfloor}(Z,g_0)}\leq\varepsilon.$$

Finally, we have the complicated definition of a cap. The last conditions essentially say that the diameter, volume, and curvature differences are controlled and are technical conditions needed in some arguments.

Definition 77 *Let (M,g) be a Riemannian 3-manifold. A (C,ε)-cap in (M,g) is a noncompact submanifold $(\mathcal{C},\, g|_{\mathcal{C}})$ together with an open submanifold $M\subset\mathcal{C}$ with the following properties:*

1. *$\mathcal{C}$ is diffeomorphic to an open 3-ball or to a punctured $\mathbb{RP}^3$.*

2. *N is a ε-neck.*

3. *$\bar{Y}=\mathcal{C}\backslash N$ is a compact submanifold with boundary. Its interior Y is called the core of $\mathcal{C}$.*

4. *The scalar curvature $R(y)>0$ for every $y\in\mathcal{C}$ and*
$$\operatorname{diam}(\mathcal{C},\, g|_{\mathcal{C}})<\frac{C}{\sqrt{\sup_{y\in\mathcal{C}}R(y)}}.$$

5.
$$\sup_{x,y\in\mathcal{C}}\frac{R(x)}{R(y)}<C.$$

6.
$$V(\mathcal{C})<\frac{C}{\left(\sup_{y\in\mathcal{C}}R(y)\right)^{3/2}}.$$

7. *For any $y\in Y$, let r_y defined by the the condition that*
$$\sup_{y'\in B(y,r_y)}R(y')=\frac{1}{r_y^2}.$$

   Then for each $y\in Y$ the ball $B(y,r_y)$ has compact closure in $\mathcal{C}$ and
$$\frac{1}{C}<\inf_{y\in Y}\frac{V(B(y,r_y))}{r_y^3}.$$

8.

$$\sup_{y\in\mathcal{C}} \frac{|\nabla R(y)|}{R(y)^{3/2}} < C$$

and

$$\sup_{y\in\mathcal{C}} \frac{\left|\frac{\partial R}{\partial t}(y)\right|}{R(y)^{2}} < C$$

8.4 How surgery works

The key observations are this:

1. For every κ and every small $\varepsilon > 0$, there is a $C_1 = C_1(\varepsilon, \kappa)$ such that a κ-solution is the union of (C_1, ε) canonical neighborhoods.
2. For every small $\varepsilon > 0$, there is a $C_2 = C_2(\varepsilon)$ and a standard solution of Ricci flow with is the union of (C_2, ε) canonical neighborhoods.
3. We can do surgery on canonical neighborhoods if they are sufficiently small and positively curved.

Consider a Ricci flow which becomes singular at a time T. Fix $T^- < T$ so that there are no surgeries in the interval $[T^-, T)$. By the assumptions, there is an open set $\Omega \subset M$ such that the curvature is bounded for all $t \in [T^-, T)$, so there is a limiting metric on Ω as $t \to T$. Every end is the end of a canonical neighborhood, which looks like a tube. We call these ends ε-horns. We can then fix a constant ρ and consider the subset $\Omega_\rho \subset \Omega$ in which the scalar curvature is bounded above by ρ^{-2}. One can then show that ε horns with boundaries in Ω_ρ are δ-necks. We then do surgery on these δ-necks by gluing in a standard solution.

8.5 Some things to prove

Here are some things we will need to do in order for this procedure to work:

1) Derive a description of κ-solutions. This will be with regard to what the asymptotic shrinking soliton is, and so we will need to show that there is one.

2) Show that solutions have canonical neighborhoods.

3) Describe the canonical solution

4) Show finite time extinction.

Chapter 9

Reduced distance

9.1 Introduction

We first wish to show that a κ-solution has a limit which is a gradient-shrinking soliton. To do this, we will need to introduce a new notion, the reduced distance.

9.2 Short discussion of W vs reduced distance

We were able to show quite a bit using the W functional, so why introduce the reduced distance (whatever that is)? Let's first think a bit about the case of Euclidean space. Recall that in Euclidean space, W becomes

$$W(M, g, u, \tau) = \int \left[\tau \frac{|\nabla u|^2}{u^2} - u \log u \right] dx - \frac{d}{2} \log(4\pi\tau) - d$$

and that W is minimized for

$$u_*(x, \tau) = (4\pi\tau)^{-d/2} \exp\left(- |x|^2 / (4\tau) \right).$$

We see that this formula essentially gives the distance function $d(x,0)^2 = |x|^2$ by

$$\frac{|x|^2}{4\tau} = -\log u_*(x, \tau) - \frac{d}{2} \log(4\pi\tau).$$

However, we have very little control over this function. One can also derive the distance function as follows:

$$d(x, 0) = \inf_{\substack{\gamma(0)=0 \\ \gamma(a)=x}} \int_0^a |\dot{\gamma}| \, d\tau.$$

This strong relationship is only true in Euclidean space. In general, there are two different concepts: the heat kernel $u_*(x, \tau)$ and the Riemannian distance function $d(x, x_0)$. The function u_* is essentially defined as the solution to a PDE and the distance function is defined by minimizing over paths. Thus, often $d(x, x_0)$ is easier to work with and easier to get more precise information about. The two things are closely related, but not the same concept. We will try to do something similar for the functional W.

9.3 $\mathcal{L}$-length and reduced distance

Consider a curve $\gamma : [0, \tau] \to M$. One can define the length of a curve as

$$\ell(\gamma) = \int_0^\tau |\dot{\gamma}(s)|\, ds$$

and the energy as

$$E(\gamma) = \frac{1}{2}\int_0^\tau |\dot{\gamma}(s)|^2\, ds.$$

Note that the length is independent of reparametrization, but the energy is not. These notions give rise to the function r_p which is the function representing the distance to the point p, i.e.,

$$r_p(x) = d(x, p).$$

It turns out that the distance function satisfies some differential equations and inequalities, in particular

$$\triangle r \leq \frac{d-1}{r}$$

for Ricci nonnegative. Note that if $r(x) = \sqrt{\sum (x^i)^2}$, then $\triangle r = \frac{d-1}{r}$. We will show this inequality later in this lecture, but for now, let's assume this and see the implications on the volume. Notice that

$$\frac{d}{dR} V(B(p, R)) = A(S(p, R))$$

where $S(p, R)$ is the geodesic sphere of radius R centered at p, and A is the area ($d-1$ dimensional volume measure) function. By Gauss lemma, we have that

$$|\nabla r| = 1$$

and also that we can decompose the metric to be

$$dr^2 + g_{S(p,r)}$$

where $g_{S(p,r)}$ is the metric on the geodesic sphere. Denote the measure on the sphere of radius r as $dA_{S(p,r)}$. We see that

$$\begin{aligned}
\int_{B(p,R)} \triangle r\, dV &= \int_{S(p,R)} (\nabla r \cdot n)\, dA_{S(p,r)} \\
&= \int_{S(p,R)} |\nabla r|\, dA_{S(p,r)} \\
&= A(S(p, R)).
\end{aligned}$$

The decomposition above implies that the volume measure can be decomposed as

$$dV = dA_{S(p,r)}dr$$

and so

$$\begin{aligned}\frac{d}{dR}\int_{B(p,R)} \triangle r dV &= \frac{d}{dR}\int_0^R \int_{S(p,r)} \triangle r dA_{S(p,r)}dr \\ &= R^{d-1}\int_{S(p,R)} \triangle r dA_{S(p,R)}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{d}{dR}\frac{A\left(S\left(p,R\right)\right)}{R^{d-1}} &= \frac{1}{R^{2(d-1)}}\left[R^{d-1}\frac{d}{dR}\int_{B(p,R)} \triangle r dV - \left(d-1\right)R^{d-2}A\left(S\left(p,R\right)\right)\right] \\ &= \frac{1}{R^{2(d-1)}}\left[R^{d-1}\int_{S(p,R)} \triangle r dA_{S(p,R)} - \left(d-1\right)R^{d-2}A\left(S\left(p,R\right)\right)\right] \\ &\leq \frac{1}{R^{2(d-1)}}\left[\left(d-1\right)R^{d-2}\int_{S(p,R)} dA_{S(p,R)} - \left(d-1\right)R^{d-2}A\left(S\left(p,r\right)\right)\right] \\ &= 0.\end{aligned}$$

This implies an inequality on volume ratios.

Theorem 78 (Bishop-Gromov theorem, simplified version) *If* $\mathrm{Rc} \geq 0$ *then* $V\left(B\left(p,r\right)\right)/r^d$ *is a nonincreasing function of* r.

Proof. We have already showed that

$$\frac{d}{dR}\frac{A\left(S\left(p,R\right)\right)}{R^{d-1}} \leq 0.$$

We now consider $0 \leq r_1 \leq r_2$. We have

$$\begin{aligned}\frac{V\left(B\left(p,r_2\right)\right) - V\left(B\left(p,r_1\right)\right)}{r_2^d - r_1^d} &= \frac{\int_{r_1}^{r_2} A\left(S\left(p,r\right)\right)dr}{\int_{r_1}^{r_2} d\ r^{d-1}dr} \\ &= \frac{\int_{r_1}^{r_2} \frac{A(S(p,r))}{r^{n-1}} r^{d-1}dr}{\int_{r_1}^{r_2} d\ r^{d-1}dr} \\ &\leq \frac{A\left(S\left(p,r_1\right)\right)}{d\ r_1^{d-1}}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{V(B(p,r_1))}{r_1^d} &= \frac{\int_0^{r_1} A(S(p,r))\,dr}{\int_0^{r_1} d\ r^{d-1}dr} \\ &= \frac{\int_0^{r_1} \frac{A(S(p,r))}{r^{n-1}} r^{d-1}dr}{\int_0^{r_1} d\ r^{d-1}dr} \\ &\geq \frac{A(S(p,r_1))}{d\ r_1^{d-1}}.\end{aligned}$$

Thus we have

$$\frac{V(B(p,r_1))}{r_1^d} \geq \frac{V(B(p,r_2)) - V(B(p,r_1))}{r_2^d - r_1^d},$$

which implies

$$\begin{aligned}0 &\leq \frac{V(B(p,r_1))}{r_1^d} - \frac{V(B(p,r_2)) - V(B(p,r_1))}{r_2^d - r_1^d} \\ &= \frac{r_2^d V(B(p,r_1)) - r_1^d V(B(p,r_2))}{r_1^d\left(r_2^d - r_1^d\right)} \\ &= \frac{r_2^d}{\left(r_2^d - r_1^d\right)}\left[\frac{V(B(p,r_1))}{r_1^d} - \frac{V(B(p,r_2))}{r_2^d}\right].\end{aligned}$$

■

We now consider Ricci flows. Let $(M, g(\tau))$ be a solution to backwards Ricci flow, i.e.,

$$\frac{\partial g}{\partial \tau} = \mathrm{Rc}(g(\tau))$$

and let $\gamma : [0, \tau] \to M$ be a curve in M. We define the $\mathcal{L}$-distance to be

$$\mathcal{L}(\gamma) = \int_0^\tau \sqrt{\sigma}\left(\left|\gamma'(\sigma)\right|^2_{g(\sigma)} + R_{g(\sigma)}\right) d\sigma.$$

This may need a bit of explanation. First, $\gamma' = \frac{d\gamma}{d\tau}$ is the tangent vector to the curve γ. Note that it is measured at σ by the metric $g(\sigma)$ (so at different ^{s}, it is measured using different metrics!) The first term looks almost like the term in the energy $\int |\gamma'|^2 d\sigma$, which can be used to derive geodesics on a Riemannian manifold. However, the addition of the term $\sqrt{\sigma}$ makes the integral scale like length, not energy (think about this).

Remark 79 *Tao uses $-\tau$ in some places because of the understanding that $\tau = -t$ and he wants $g(t)$ to be a solution to Ricci flow and $V(t)$ (defined later) to be defined on t. We will allow g to be parametrized by τ and so there is no need for this.*

In the smooth, fixed manifold case, one considers the length functional

$$L(\gamma) = \int_0^{\tau_1} \left|\gamma'(\sigma)\right|_g d\sigma$$

and then one can find the distance $d(x_0, x)$ between points by taking the infimum of length over all curves connecting those two points. One can define the distance function $r_{x_0}(x)$ which is the function which returns the distance to a fixed point x_0. The analogue of this in the Ricci flow case is the ℓ-distance (also called reduced length):

$$\ell_{(0,x_0)}(\tau, x) = \frac{1}{2\sqrt{\tau}} \inf\{\mathcal{L}(\gamma)\}$$

where the inf is over all paths γ from x_0 to x.

Remark 80 *Note there is also the $\mathcal{L}$-distance, which is $2\sqrt{\tau}\ell$.*

One can finally define the reduced volume

$$\tilde{V}_{(0,x_0)}(\tau) = \int_M \tau^{-d/2} \exp\left[-\ell_{(0,x_0)}(\tau, x)\right] dV_{g(\tau)}.$$

Our main goal will be to show the following:

Theorem 81

$$\frac{\partial}{\partial \tau}\ell_{(0,x_0)} - \triangle_{g(\tau)}\ell_{(0,x_0)} + \left|\nabla \ell_{(0,x_0)}\right|^2_{g(\tau)} - R + \frac{d}{2\tau} \geq 0.$$

As a corollary, we get monotonicity of the reduced volume if $\frac{\partial}{\partial \tau} g = 2\operatorname{Rc}(g)$.

Corollary 82 (Reduced Volume is monotone)

$$\frac{\partial}{\partial \tau}\tilde{V}_{(0,x_0)}(\tau) \leq 0.$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial \tau}\tilde{V}_{(0,x_0)}(\tau) &= \frac{\partial}{\partial \tau}\int_M \tau^{-d/2} \exp\left[-\ell_{(0,x_0)}(\tau, x)\right] dV_{g(\tau)} \\
&= -\frac{d}{2\tau}\tilde{V}_{(0,x_0)}(\tau) - \int_M \frac{\partial \ell_{(0,x_0)}}{\partial \tau}\tau^{-d/2} \exp\left[-\ell_{(0,x_0)}(\tau, x)\right] dV_{g(\tau)} \\
&+ \int_M \tau^{-d/2} \exp\left[-\ell_{(0,x_0)}(\tau, x)\right] R dV_{g(\tau)} \\
&\leq -\frac{d}{2\tau}\tilde{V}_{(0,x_0)}(\tau) + \int_M \tau^{-d/2} \exp\left[-\ell_{(0,x_0)}(\tau, x)\right] R dV_{g(\tau)} \\
&+ \int_M \left(-\triangle_{g(\tau)}\ell_{(0,x_0)} + |\nabla \ell|^2_{g(\tau)} - R + \frac{d}{2\tau}\right)\tau^{-d/2} \exp\left[-\ell_{(0,x_0)}(\tau, x)\right] dV_{g(\tau)} \\
&= 0.
\end{aligned}$$

■

Of course, all of this assumes sufficient regularity on ℓ, which is not, in general true. However, this argument can be made rigorous in some generality, including past the "conjugate radius."

9.4 Variations of length and the distance function

We start with the energy functional given above. We can do calculus of variations as follows. Consider a variation $\Gamma(t,s)$ such that

$$\Gamma(t,s=0) = \gamma(t)$$
$$\frac{\partial}{\partial s}\Gamma(t,s=0) = X(t),$$

so we consider the variation to be X. Furthermore, we consider variations which fix the endpoints, i.e., $X(0) = X(\tau) = 0$. Compute

$$\begin{aligned}\delta E_\gamma(X) &= \delta\frac{1}{2}\int_0^\tau g(\dot\gamma,\dot\gamma)\,dt \\ &= \frac{1}{2}\left.\frac{\partial}{\partial s}\right|_{s=0}\int_0^\tau g\left(\frac{\partial\Gamma}{\partial t},\frac{\partial\Gamma}{\partial t}\right)dt \\ &= \int_0^\tau g\left(D_s\frac{\partial\Gamma}{\partial t},\frac{\partial\Gamma}{\partial t}\right)dt \\ &= \int_0^\tau g(D_tX,\dot\gamma)\,dt \\ &= g(X,\dot\gamma)|_0^\tau - \int_0^\tau g(X,D_t\dot\gamma)\,dt.\end{aligned}$$

Remark 83 *The notion D_tX means $D_tX = \nabla_{\frac{\partial}{\partial t}}X$. It only depends on the values along the curve γ. (Why?)*

At a critical point for the energy, we must have $\delta E_\gamma(X) = 0$ for all X, so we get that $D_t\dot\gamma = 0$ if the endpoints are fixed, i.e., if $X(0) = X(\tau) = 0$. This is the geodesic equation. Notice that if we restrict to a geodesic curve, (i.e., $D_t\dot\gamma = 0$) and fix the initial point ($X(0) = 0$), then

$$\delta E_\gamma(X) = g(X(\tau),\dot\gamma(\tau)).$$

On a geodesic,

$$\frac{d}{dt}|\dot\gamma(t)|^2 = 2g(D_t\dot\gamma,\dot\gamma) = 0$$

so γ has constant velocity. So along a geodesic, we have

$$E(\gamma) = \frac{1}{2}\int_0^\tau |\dot\gamma|^2\,ds = \frac{1}{2}|\dot\gamma|^2\tau$$
$$\ell(\gamma) = \int_0^\tau |\dot\gamma|\,ds = |\dot\gamma|\,\tau$$

and so if γ is a geodesic, then

$$E(\gamma) = \frac{1}{2\tau}[\ell(\gamma)]^2$$

and so we have that if γ is the gives the distance $d(p,x)$, then

$$E(\gamma) = \frac{1}{2\tau} d(p,x)^2 .$$

If we want to compute a variation

$$\frac{d}{ds} d(p, x(s))^2$$

away from the cut locus (so variations in geodesics stay minimizing), we have

$$\frac{d}{ds} d(p,x(s))^2 = 2\tau \delta E_\gamma (X) = 2\tau g\left(\frac{dx}{ds}, \dot{\gamma}(\tau)\right),$$

where X is a variation of geodesics (one must show that these exist, but they do!) Note that we can always reparametrize so that $\tau = 1$, and then we get that

$$\left|\nabla d(p,x)^2\right|^2 = 4\left|\dot{\gamma}(1)\right|^2 ,$$

or

$$4d(p,x)^2 \left|\nabla d(p,x)\right|^2 = 4\left|\dot{\gamma}(1)\right|^2 = 4d(p,x)^2$$

and so

$$\left|\nabla d(p,x)\right| = 1.$$

Now compute the second variation of energy when γ is a geodesic. We get

$$\begin{aligned}
\delta^2 E_\gamma (X,X) &= \frac{\partial}{\partial s}\int_0^\tau g(D_t X, \dot{\gamma})\, dt \\
&= g(X,\dot{\gamma})|_0^\tau - \int_0^\tau g(X, D_t\dot{\gamma})\, dt.
\end{aligned}$$

$$\begin{aligned}
\delta^2 E_\gamma (X,X) &= \frac{1}{2}\frac{\partial^2}{\partial s^2}\bigg|_{s=0}\int_0^\tau g\left(\frac{\partial\Gamma}{\partial t},\frac{\partial\Gamma}{\partial t}\right) dt \\
&= \int_0^\tau g\left(D_s D_s \frac{\partial\Gamma}{\partial t},\frac{\partial\Gamma}{\partial t}\right) + g\left(D_s\frac{\partial\Gamma}{\partial t}, D_s\frac{\partial\Gamma}{\partial t}\right) dt \\
&= \int_0^\tau g\left(D_s D_t \frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right) + g\left(D_t\frac{\partial\Gamma}{\partial s}, D_t\frac{\partial\Gamma}{\partial s}\right) dt \\
&= \int_0^\tau g\left(D_t D_s \frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right) + g\left(R\left(\frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right)\frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right) + g(D_t X, D_t X)\, dt. \\
&= g(\nabla_X X, \gamma)|_0^\tau + \int_0^\tau g(R(X,\dot{\gamma})X,\dot{\gamma}) + g(D_t X, D_t X)\, dt
\end{aligned}$$

The first term is zero if the variation is a variation of geodesics at the endpoints (or fixed endpoints). Note our only assumptions are this and that γ is a geodesic.

Let's again parametrize our geodesic between 0 and 1 and fix $\gamma(0)$ and vary the other endpoint $x(s)$ so that $X(t) = \frac{\partial \Gamma}{\partial s}(t,0)$ "varies through geodesics." Then

$$\frac{1}{2}\frac{d^2}{ds^2} d(p,x(s))^2 = \int_0^1 g(R(X,\dot{\gamma})X,\dot{\gamma}) + g(D_t X, D_t X)\,dt.$$

Furthermore, we can take $X(t) = t\frac{\partial x}{\partial s}$, where $\frac{\partial x}{\partial s}$ is the parallel translation along γ of $\frac{\partial x}{\partial s}$.

Remark 84 *The parallel translation of a vector v along a curve γ is the solution to the system of ODE*

$$D_t v = \nabla_{\gamma'} v = \frac{\partial v^k}{\partial t} + \Gamma_{ij}^k (\gamma')^i v^j = 0.$$

Then we get

$$\frac{1}{2}\frac{d^2}{ds^2} d(p,x(s))^2 = \int_0^1 t^2 g\left(R\left(\frac{\partial x}{\partial s},\dot{\gamma}\right)\frac{\partial x}{\partial s},\dot{\gamma}\right)dt + g\left(\frac{\partial x}{\partial s},\frac{\partial x}{\partial s}\right).$$

Thus if we sum over an orthonormal frame and assume that the sectional curvature is nonnegative, we have that

$$\frac{1}{2}\triangle d(p,x)^2 \leq d.$$

And so

$$\frac{1}{2}\triangle\left(f^2\right) = f\triangle f + |\nabla f|^2$$

and so

$$\triangle d(p,x) \leq \frac{d-1}{d(p,x)}$$

Here's another derivation: We now know that for a curve $x(s)$,

$$\frac{1}{2} d(x_0,x(s))^2 = E(\gamma_s)$$

where $\gamma_s(t)$ are curves such that

$$D_t \gamma_s' = \nabla_{\gamma_s'}\gamma_s' = 0$$

for all s. We may parametrize all curves between 0 and 1. Furthermore, we know that if we consider the variation $\Gamma(s,t) = \gamma_s(t)$, then

$$\begin{aligned}\left.\frac{\partial}{\partial s}\right|_{s=0}\gamma_s(t) &= X(t), \\ X(0) &= 0, \\ X(1) &= \frac{dx}{ds}.\end{aligned}$$

$$E(\gamma_s) = E(\gamma_0) + s\delta E_{\gamma_0}(X) + \frac{1}{2}s^2\delta^2 E_{\gamma_0}(X, X) + O(s^3),$$

where

$$\delta^2 E_{\gamma_0}(X, X) = \frac{d^2}{ds^2}E(\gamma_s).$$

Thus the first derivative is

$$\begin{aligned}\left.\frac{d}{ds}\right|_{s=0} \frac{1}{2}d(x_0, x(s))^2 &= \lim_{s\to 0} \frac{\frac{1}{2}d(x_0, x(s))^2 - \frac{1}{2}d(x_0, x(0))^2}{s} \\ &= \delta E_{\gamma_0}(X).\end{aligned}$$

The key fact is that under such a variation, we have

$$\delta E_{\gamma_0}(X) = g(\gamma_0'(1), X(1)) = g\left(\gamma_0'(1), \frac{dx}{ds}(0)\right)$$

so we did not need to know much about $X(s)$ in general!

Now, to compute the second derivative, we look at

$$\begin{aligned}\left.\frac{d^2}{ds^2}\right|_{s=0} \frac{1}{2}d(x_0, x(s))^2 &= \lim_{s\to 0} \frac{\frac{1}{2}d(x_0, x(s))^2 + \frac{1}{2}d(x_0, x(-s))^2 - 2\frac{1}{2}d(x_0, x(0))^2}{s^2} \\ &= \delta^2 E_{\gamma_0}(X, X).\end{aligned}$$

$$\begin{aligned}\delta^2 E_\gamma(X, X) &= \frac{1}{2}\left.\frac{\partial^2}{\partial s^2}\right|_{s=0} \int_0^1 g\left(\frac{\partial\Gamma}{\partial t}, \frac{\partial\Gamma}{\partial t}\right) dt \\ &= \int_0^1 g\left(D_s D_s \frac{\partial\Gamma}{\partial t}, \frac{\partial\Gamma}{\partial t}\right) + g\left(D_s\frac{\partial\Gamma}{\partial t}, D_s\frac{\partial\Gamma}{\partial t}\right) dt \\ &= \int_0^1 g\left(D_s D_t \frac{\partial\Gamma}{\partial s}, \frac{\partial\Gamma}{\partial t}\right) + g\left(D_t\frac{\partial\Gamma}{\partial s}, D_t\frac{\partial\Gamma}{\partial s}\right) dt \\ &= \int_0^1 g\left(D_t D_s \frac{\partial\Gamma}{\partial s}, \frac{\partial\Gamma}{\partial t}\right) + g\left(R\left(\frac{\partial\Gamma}{\partial s}, \frac{\partial\Gamma}{\partial t}\right)\frac{\partial\Gamma}{\partial s}, \frac{\partial\Gamma}{\partial t}\right) + g(D_t X, D_t X)\, dt. \\ &= g\left(D_s\frac{\partial\Gamma}{\partial s}, \frac{\partial\Gamma}{\partial t}\right)\Big|_0^1 + \int_0^1 g(R(X, \gamma')X, \gamma') + g(D_t X, D_t X)\, dt \\ &= -\int_0^1 K(X, \gamma')|X|^2|\gamma'|^2 + \int_0^1 g(D_t X, D_t X)\, dt\end{aligned}$$

Since I can also choose the variation so that $D_s\frac{\partial\Gamma}{\partial s}(s, t=1) = 0$.

Fact: The vector field $X = tv$ where v is the parallel transport of $X(1) = \frac{\partial x}{\partial s}$ along the curve γ. This implies that

$$\begin{aligned}\delta^2 E_\gamma(X, X) &= -\int_0^1 K(X, \gamma')|X|^2|\gamma'|^2 + \int_0^1 g(D_t X, D_t X)\, dt \\ &= -\int_0^1 K(X, \gamma')|X|^2|\gamma'|^2 + \left|\frac{\partial x}{\partial s}\right|^2.\end{aligned}$$

Now if we take $\left|\frac{\partial x}{\partial s}\big|_{s=0}\right| = 1$, then we have

$$\frac{d^2}{ds^2}\bigg|_{s=0} \frac{1}{2} d\left(x_0, x\left(s\right)\right)^2 \leq \left|\frac{\partial x}{\partial s}\right|^2 = 1$$

if the sectional curvatures are positive. Now take normal coordinates at x. At the center of normal coordinates,

$$\triangle f\left(0\right) = \left(\frac{\partial}{\partial x^1}\right)^2 f + \cdots \left(\frac{\partial}{\partial x^d}\right)^2 f.$$

So, taking $x\left(s\right) = \exp_x\left(s\frac{\partial}{\partial x^i}\right)$, we conclude that

$$\triangle \left[\frac{1}{2} d\left(x_0, x\right)^2\right] \leq d.$$

Furthermore, we have

$$\begin{aligned} \triangle \left[\frac{1}{2} d\left(x_0, x\right)^2\right] &= \nabla \cdot \left(d\left(x_0, x\right) \nabla d\left(x_0, x\right)\right) \\ &= d\left(x_0, x\right) \triangle d\left(x_0, x\right) + \left|\nabla d\left(x_0, x\right)\right|^2 \\ &= d\left(x_0, x\right) \triangle d\left(x_0, x\right) + 1. \end{aligned}$$

Thus, we get

$$\begin{aligned} d\left(x_0, x\right) \triangle d\left(x_0, x\right) + 1 &\leq d \\ \triangle d\left(x_0, x\right) &\leq \frac{d-1}{d\left(x_0, x\right)}. \end{aligned}$$

9.5 Variations of the reduced distance

We will do the same thing to the reduced distance. Consider

$$\mathcal{L}\left(\gamma\right) = \int_0^\tau \sqrt{\sigma}\left(\left|\gamma'\left(\sigma\right)\right|^2_{g(\sigma)} + R_{g(\sigma)}\right) d\sigma.$$

Actually, we want to think of this as a functional on spacetime paths. A spacetime path γ is a map

$$\gamma : [0, \tau] \to M \times I$$

where I is an interval. We will only consider paths that look like

$$\gamma\left(s\right) = \left(\tilde{\gamma}\left(s\right), s\right). \tag{9.1}$$

We can naturally define $R\left(\gamma\left(s\right)\right) = R_{g(s)}\left(\tilde{\gamma}\left(s\right)\right)$. We then have the formulation of the functional as

$$\mathcal{L}\left(\gamma\right) = \int_0^\tau \sqrt{\sigma}\left(\left|\tilde{\gamma}'\left(\sigma\right)\right|^2_{g(\sigma)} + R\left(\gamma\left(\sigma\right)\right)\right) d\sigma.$$

We will define the reduced length as

$$\ell_{(0,x_0)}(\tau, x) = \frac{1}{2\sqrt{\tau}} \inf \{\mathcal{L}(\gamma) : \gamma \text{ are of the form } (9.1)\}.$$

Note that the infimum can also be considered as the infimum over all paths $\tilde{\gamma}$ on M.

Note that the tangent space of $M \times I$ splits as $TM \times TI$ and TI is spanned by $\frac{\partial}{\partial \tau}$. There is a notion of a horizontal vector field, which is a vector field X such that $d\tau(X) = 0$. Since variations of paths of the form (9.1) must be horizontal vector fields, we first consider the first variation of $\mathcal{L}$ with respect to a horizontal vector field X. Note that $\tilde{\gamma}' = \gamma' - \frac{\partial}{\partial \tau}$ is a horizontal vector field (but γ' is not).

Remark 85 *Horizontal vector fields on $M \times I$ are in one-to-one correspondence with vector fields on M. For this reason, we may abuse notation and use the same notation for both vector fields. In particular, $\tilde{\gamma}'$ will be considered both a vector field on M and a horizontal vector field on $M \times I$.*

We take a variation of $\Gamma(s, \sigma)$ such that

$$\begin{aligned}
\Gamma(0, \sigma) &= \gamma(\sigma) = (\tilde{\gamma}(\sigma), \sigma) \\
\Gamma(s, \sigma) &= \left(\tilde{\Gamma}(s, \sigma), \sigma\right) \\
\left.\frac{\partial}{\partial s}\right|_{s=0} \Gamma(s, \sigma) &= X.
\end{aligned}$$

This is a horizontal variation. We get

$$\delta\mathcal{L}_\gamma(X) = \left.\frac{\partial}{\partial s}\right|_{s=0} \delta\mathcal{L}(\Gamma) = \int_0^\tau \sqrt{\sigma}\left(2\left\langle \nabla_{\frac{\partial \Gamma}{\partial s}} \frac{\partial \tilde{\Gamma}}{\partial \sigma}, \frac{\partial \tilde{\Gamma}}{\partial \sigma}\right\rangle_{g(\sigma)} + \nabla_{\frac{\partial \Gamma}{\partial s}} R_{g(\sigma)}\right) d\sigma.$$

We do the same thing we did with the energy functional, finding

$$\begin{aligned}
\delta\mathcal{L}_\gamma(X) &= \left.\int_0^\tau \sqrt{\sigma}\left(2\left\langle \nabla_{\frac{\partial \tilde{\Gamma}}{\partial \sigma}} \frac{\partial \Gamma}{\partial s}, \frac{\partial \tilde{\Gamma}}{\partial \sigma}\right\rangle_{g(\sigma)} + \nabla_{\frac{\partial \Gamma}{\partial s}} R_{g(\sigma)}\right) d\sigma\right|_{s=0} \\
&= \left.\int_0^\tau \sqrt{\sigma}\left(2\frac{d}{d\sigma}\left\langle \frac{\partial \Gamma}{\partial s}, \frac{\partial \tilde{\Gamma}}{\partial \sigma}\right\rangle_{g(\sigma)} - 4\,\mathrm{Rc}\left(\frac{\partial \Gamma}{\partial s}, \frac{\partial \tilde{\Gamma}}{\partial \sigma}\right) - 2\left\langle \frac{\partial \Gamma}{\partial s}, \nabla_{\frac{\partial \tilde{\Gamma}}{\partial \sigma}} \frac{\partial \tilde{\Gamma}}{\partial \sigma}\right\rangle_{g(\sigma)} + \nabla_{\frac{\partial \Gamma}{\partial s}} R_{g(\sigma)}\right) d\sigma\right|_{s=0} \\
&= \int_0^\tau \sqrt{\sigma}\left(2\frac{d}{d\sigma}\langle X, \tilde{\gamma}'\rangle_{g(\sigma)} - 4\,\mathrm{Rc}(X, \tilde{\gamma}') - 2\langle X, \nabla_{\tilde{\gamma}'}\tilde{\gamma}'\rangle_{g(\sigma)} + \nabla_X R_{g(\sigma)}\right) d\sigma
\end{aligned}$$

since if X and Y are horizontal,

$$\frac{d}{d\tau} g(X, Y) = 2\,\mathrm{Rc}(X, Y) + g(\nabla_{\tilde{\gamma}'} X, Y) + g(X, \nabla_{\tilde{\gamma}'} Y).$$

Remark 86 *In general, the formula is*

$$\frac{d}{d\tau}g(X,Y)=\frac{\partial}{\partial\tau}g(X,Y)+\nabla_{\tilde{\gamma}'}g(X,Y),$$

which has more terms since $\nabla_{\tilde{\gamma}'}X\neq 0$, etc.

Remark 87 *I used the notation of total derivative since there is the variation of the metric with respect to the time parameter of Ricci flow and also the variation with respect to γ'. I have also used $'$ instead of dot to denote derivative with respect to τ or σ.*

Now we need to take the $\frac{d}{d\sigma}$ out, so we need that

$$\frac{d}{d\sigma}\left(2\sqrt{\sigma}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)}\right)=\frac{1}{\sqrt{\sigma}}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)}+2\sqrt{\sigma}\frac{d}{d\sigma}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)},$$

so

$$\begin{aligned}\delta\mathcal{L}_{\gamma}(X)=&\int_0^{\tau}\frac{d}{d\sigma}\left(2\sqrt{\sigma}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)}\right)-\frac{1}{\sqrt{\sigma}}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)}\\&+\sqrt{\sigma}\left(-4\operatorname{Rc}(X,\tilde{\gamma}')-2\left\langle X,\nabla_{\tilde{\gamma}'}\tilde{\gamma}'\right\rangle_{g(\sigma)}+\nabla_X R_{g(\sigma)}\right)d\sigma\\=&\ 2\sqrt{\sigma}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)}\Big|_0^{\tau}-2\sqrt{\sigma}\int_0^{\tau}\left\langle G,X\right\rangle_{g(\sigma)}d\sigma\end{aligned}$$

where G is the vector field

$$G(\sigma)=\nabla_{\tilde{\gamma}'}\tilde{\gamma}'+\frac{1}{2\sigma}\tilde{\gamma}'+2\operatorname{Rc}(\tilde{\gamma}')-\frac{1}{2}\nabla R_{g(\sigma)}$$

where $\operatorname{Rc}(X)$ is the vector field on M such that

$$\left\langle\operatorname{Rc}(X),Y\right\rangle=\operatorname{Rc}(X,Y)$$

for all for all vector fields Y on M. Notice that G does not depend on the variation X. Note that if γ is a minimizer with fixed endpoints (i.e., for all variations X such that $X(0)=X(\tau)=0$), then we must have that $G=0$. This is the $\mathcal{L}$-geodesic equation.

Problem 88 *If we are in Euclidean space, what are the $\mathcal{L}$-geodesics?*

Now supposing there is a unique minimizer and that ℓ is a smooth function, we can consider variations through $\mathcal{L}$-geodesics such that $X(0)$ is fixed to get

$$\delta\mathcal{L}_{\gamma}(X)=2\sqrt{\tau}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\tau)},$$

which implies that

$$\frac{\partial}{\partial s}\ell_{(0,x_0)}(\tau,x(s))=\left\langle\frac{\partial x}{\partial s},\tilde{\gamma}'\right\rangle_{g(\tau)}$$

(recall that ℓ divides by $\sqrt{\tau}$). This can also be written as

$$\nabla \ell_{(0,x_0)} = \tilde{\gamma}'.$$

Note that the reduced length also depends on time, so let's compute the time derivative as well:

$$\delta \mathcal{L}_\gamma \left(\frac{d}{d\tau} \right) = \sqrt{\tau} \left(\left| \tilde{\gamma}'(\tau) \right|^2_{g(\tau)} + R_{g(\tau)} \right).$$

Note that this need not be zero since we are not minimizing in this direction. We find that

$$\frac{d}{d\tau} \left(2\sqrt{\tau} \ell_{(0,x_0)} \right) = \sqrt{\tau} \left(\left| \tilde{\gamma}'(\tau) \right|^2_{g(\tau)} + R_{g(\tau)} \right).$$

Now, the total derivative decomposes as

$$\frac{d}{d\tau} \ell_{(0,x_0)} = \frac{\partial}{\partial \tau} \ell_{(0,x_0)} + \nabla_{\tilde{\gamma}'} \ell_{(0,x_0)},$$

and so

$$\frac{\partial}{\partial \tau} \ell_{(0,x_0)}(\tau, x) = \frac{1}{2} \left(\left| \tilde{\gamma}'(\tau) \right|^2_{g(\tau)} + R_{g(\tau)} \right) - \frac{1}{2\tau} \ell_{(0,x_0)}(\tau, x) - \frac{1}{2\sqrt{\tau}} \left| \tilde{\gamma}'(\tau) \right|^2_{g(\tau)}.$$

In order to look at second derivatives of $\ell_{(0,x_0)}$, we consider variations among minimizers. In particular, we let γ be a minimizer with $G = 0$. We compute the second variation of $\mathcal{L}$. It will be convenient to assume the variation is through $\mathcal{L}$-geodesics, that is

$$\nabla_{\frac{\partial \Gamma}{\partial s}} \frac{\partial \Gamma}{\partial s} = 0.$$

This implies that

$$\left. \frac{\partial^2}{\partial s^2} \right|_{s=0} R(\Gamma(s,\sigma)) = \nabla_X \nabla_X R$$

where $\nabla_X \nabla_X R$ means the Hessian $\nabla^2 R(X,X)$. Then we have,

$$\begin{aligned}
\delta^2 \mathcal{L}_\gamma(X,X) &= \left. \frac{d^2}{ds^2} \right|_{s=0} \mathcal{L}(\Gamma(s,\sigma)) \\
&= \left. \frac{d}{ds} \right|_{s=0} \int_0^\tau \sqrt{\sigma} \left(2 \left\langle \nabla_{\frac{\partial \Gamma}{\partial s}} \frac{\partial \tilde{\Gamma}}{\partial \sigma}, \frac{\partial \tilde{\Gamma}}{\partial \sigma} \right\rangle_{g(\sigma)} + \nabla_{\frac{\partial \Gamma}{\partial s}} R_{g(\sigma)} \right) d\sigma \\
&= \int_0^\tau \sqrt{\sigma} \left(2 \left\langle \nabla_{\frac{\partial \Gamma}{\partial s}} \nabla_{\frac{\partial \Gamma}{\partial s}} \frac{\partial \tilde{\Gamma}}{\partial \sigma}, \frac{\partial \tilde{\Gamma}}{\partial \sigma} \right\rangle_{g(\sigma)} + 2 \left\langle \nabla_{\frac{\partial \Gamma}{\partial s}} \frac{\partial \tilde{\Gamma}}{\partial \sigma}, \nabla_{\frac{\partial \Gamma}{\partial s}} \frac{\partial \tilde{\Gamma}}{\partial \sigma} \right\rangle_{g(\sigma)} + \nabla_{\frac{\partial \Gamma}{\partial s}} \nabla_{\frac{\partial \Gamma}{\partial s}} R_{g(\sigma)} + \nabla_{\frac{\partial \Gamma}{\partial s}} \frac{\partial \Gamma}{\partial s}(R) \right) d\sigma \\
&= \int_0^\tau \sqrt{\sigma} \left(2 \left\langle \nabla_X \nabla_X \tilde{\gamma}', \tilde{\gamma}' \right\rangle_{g(\sigma)} + 2 \left\langle \nabla_X \tilde{\gamma}', \nabla_X \tilde{\gamma}' \right\rangle_{g(\sigma)} + \nabla_X \nabla_X R_{g(\sigma)} \right) d\sigma
\end{aligned}$$

$$= \int_0^\tau \sqrt{\sigma} \left(2 \left\langle \nabla_{\frac{\partial \Gamma}{\partial s}} \nabla_{\frac{\partial \Gamma}{\partial s}} \frac{\partial \tilde{\Gamma}}{\partial \sigma}, \frac{\partial \tilde{\Gamma}}{\partial \sigma} \right\rangle_{g(\sigma)} + 2 \left\langle \nabla_{\frac{\partial \Gamma}{\partial s}} \frac{\partial \tilde{\Gamma}}{\partial \sigma}, \nabla_{\frac{\partial \Gamma}{\partial s}} \frac{\partial \tilde{\Gamma}}{\partial \sigma} \right\rangle_{g(\sigma)} + \nabla_{\frac{\partial \Gamma}{\partial s}} \nabla_{\frac{\partial \Gamma}{\partial s}} R_{g(\sigma)} + \nabla_{\frac{\partial \Gamma}{\partial s}} \frac{\partial \Gamma}{\partial s}(R) \right) d\sigma$$

$$\begin{aligned}\delta\mathcal{L}_{\gamma}\left(X\right) = \int_0^{\tau} \frac{d}{d\sigma}\left(2\sqrt{\sigma}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)}\right) - \frac{1}{\sqrt{\sigma}}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)} \\ + \sqrt{\sigma}\left(-4\operatorname{Rc}\left(X,\tilde{\gamma}'\right) - 2\left\langle X,\nabla_{\tilde{\gamma}'}\tilde{\gamma}'\right\rangle_{g(\sigma)} + \nabla_X R_{g(\sigma)}\right) d\sigma \\ = 2\sqrt{\sigma}\left\langle X,\tilde{\gamma}'\right\rangle_{g(\sigma)}\Big|_0^{\tau} - 2\sqrt{\sigma}\int_0^{\tau}\left\langle G,X\right\rangle_{g(\sigma)} d\sigma\end{aligned}$$

Chapter 10

Problems

1) Show that if (M, g) is κ-noncollapsed at x_0 at the scale of $\sqrt{\tau}$, then it is κ-noncollapsed at x_0 at all scales smaller than $\sqrt{\tau}$.

2) Recall the functionals

$$F(M, g, f) = \int \left(R + |\nabla f|^2\right) e^{-f} dV$$
$$W(M, g, f, \tau) = \int \left[\tau\left(R + |\nabla f|^2\right) + f - d\right] (4\pi\tau)^{-d/2} e^{-f} dV$$

and their corresponding

$$\lambda(M, g) = \inf\left\{F(M, g, f) : \int_M e^{-f} dV = 1\right\}$$
$$\mu(M, g, \tau) = \inf\left\{W(M, g, f, \tau) : \int_M (4\pi\tau)^{-d/2} e^{-f} e^{-f} dV = 1\right\}.$$

By considering variations of the function f (with M, g, τ fixed), show that the minimizers f_* for λ and $f_\#$ for μ satisfy the differential equations

$$2\triangle f_* - |\nabla f_*|^2 + R = \lambda,$$
$$\tau\left(R + 2\triangle f_\# - |\nabla f_\#|^2\right) + f_\# - d = \mu.$$

Hint: you must use Lagrange multipliers to enforce the constraint.

3) Suppose $(M, g) = ([0, a] \times [0, b] / \sim, g_{flat})$ is a flat torus gotten by identifying the interval $[0, a] \times [0, b]$. Find constants c_* and $c_\#$ such that $f_* = c_*$ and $f_\# = c_\#$ satisfy both the differential equations and the constraint equations (the constants c_* and $c_\#$ should depend on the volume, which equals ab). In fact, you could do this for any closed manifold with $R = 0$.

4) On the same torus, suppose $a \leq b$. For which scales $\sqrt{\tau}$ is (M, g) a-noncollapsed? How does this compare with the estimate one might get from part 3?

Then we have

$$W(M,g,f,\tau) = \int \left[\tau |\nabla f|^2 + f - d\right] (4\pi\tau)^{-d/2} e^{-f} dV$$

Now consider

$$\mu(M,g,\tau) = \inf \left\{ \int \left[\tau |\nabla f|^2 + f\right] (4\pi\tau)^{-d/2} e^{-f} dV - d \right\}$$

And so a minimizer satisfies

$$\tau \left(R + 2\triangle f - |\nabla f|^2\right) + f - d = \mu$$

If we restrict to $R = 0$, then

$$\tau \left(2\triangle f - |\nabla f|^2\right) + f - d = \mu$$

Notice that if $f(x) = c$, then

$$\int (4\pi\tau)^{-d/2} e^{-f} dV = (4\pi\tau)^{-d/2} e^{-c} V(M)$$

and so the constraint is

$$\begin{aligned} (4\pi\tau)^{-d/2} e^{-c} V(M) &= 1 \\ c &= \log\left[(4\pi\tau)^{-d/2} V(M)\right], \end{aligned}$$

which satisfies

$$\tau \left(2\triangle f - |\nabla f|^2\right) + f - d = \log\left[(4\pi\tau)^{-d/2} V(M)\right] - d = \mu$$

or

$$\mu = \log \frac{ab}{\tau^{d/2}} - d - \frac{d}{2} \log(4\pi)$$

$$\begin{aligned} \delta F &= \int \left(-Rh - |\nabla f|^2 h + 2\nabla f \cdot \nabla h\right) e^{-f} dV \\ &= \int \left(-Rh - |\nabla f|^2 h - 2\triangle f + 2 |\nabla f|^2\right) e^{-f} dV \end{aligned}$$

$$\begin{aligned} \delta W &= \int \left[\tau (2\nabla f \cdot \nabla h) + h - h\left[\tau\left(R + |\nabla f|^2\right) + f - d\right]\right] (4\pi\tau)^{-d/2} e^{-f} dV \\ &= -\int \left[h\left[\tau\left(R + 2\triangle f - |\nabla f|^2\right) + f - d\right]\right] (4\pi\tau)^{-d/2} e^{-f} dV \end{aligned}$$