Math 538 Problems 1

Spring 2009

1) Show that if (M, g) is κ-noncollapsed at x_0 at the scale of $\sqrt{\tau}$, then it is κ-noncollapsed at x_0 at all scales smaller than $\sqrt{\tau}$.

2) Recall the functionals

$$F(M, g, f) = \int \left(R + |\nabla f|^2\right) e^{-f} dV$$
$$W(M, g, f, \tau) = \int \left[\tau \left(R + |\nabla f|^2\right) + f - d\right] (4\pi\tau)^{-d/2} e^{-f} dV$$

and their corresponding

$$\lambda(M, g) = \inf \left\{F(M, g, f) : \int_M e^{-f} dV = 1\right\}$$
$$\mu(M, g, \tau) = \inf \left\{W(M, g, f, \tau) : \int_M (4\pi\tau)^{-d/2} e^{-f} e^{-f} dV = 1\right\}.$$

By considering variations of the function f (with M, g, τ fixed), show that the minimizers f_* for λ and $f_\#$ for μ satisfy the differential equations

$$2\triangle f_* - |\nabla f_*|^2 + R = \lambda,$$
$$\tau \left(R + 2\triangle f_\# - |\nabla f_\#|^2\right) + f_\# - d = \mu.$$

Hint: you must use Lagrange multipliers to enforce the constraint.

3) Suppose $(M, g) = ([0, a] \times [0, b] / \sim, g_{flat})$ is a flat torus gotten by identifying the interval $[0, a] \times [0, b]$. Find constants c_* and $c_\#$ such that $f_* = c_*$ and $f_\# = c_\#$ satisfy both the differential equations and the constraint equations (the constants c_* and $c_\#$ should depend on the volume, which equals ab). In fact, you could do this for any closed manifold with $R = 0$.

4) On the same torus, suppose $a \leq b$. For which scales $\sqrt{\tau}$ is (M, g) a-noncollapsed? How does this compare with the estimate one might get from part 3 and our work relating log-Sobolev inequalities to noncollapse?