Math 538 Problems 1

Spring 2009

1) Show that if \((M, g)\) is \(\kappa\)-noncollapsed at \(x_0\) at the scale of \(\sqrt{\tau}\), then it is \(\kappa\)-noncollapsed at \(x_0\) at all scales smaller than \(\sqrt{\tau}\).

2) Recall the functionals

\[
F(M, g, f) = \int \left( R + |\nabla f|^2 \right) e^{-f} dV
\]

\[
W(M, g, f, \tau) = \int \left[ \tau \left( R + |\nabla f|^2 \right) + f - d \right] (4\pi\tau)^{-d/2} e^{-f} dV
\]

and their corresponding

\[
\lambda(M, g) = \inf \left\{ F(M, g, f) : \int_M e^{-f} dV = 1 \right\}
\]

\[
\mu(M, g, \tau) = \inf \left\{ W(M, g, f, \tau) : \int_M (4\pi\tau)^{-d/2} e^{-f} dV = 1 \right\}.
\]

By considering variations of the function \(f\) (with \(M, g, \tau\) fixed), show that the minimizers \(f_*\) for \(\lambda\) and \(f_\#\) for \(\mu\) satisfy the differential equations

\[
2\Delta f_* - |\nabla f_*|^2 + R = \lambda,
\]

\[
\tau \left( R + 2\Delta f_\# - |\nabla f_\#|^2 \right) + f_\# - d = \mu.
\]

Hint: you must use Lagrange multipliers to enforce the constraint.

3) Suppose \((M, g) = ([0, a] \times [0, b] / \sim, g_{flat})\) is a flat torus gotten by identifying the interval \([0, a] \times [0, b]\). Find constants \(c_*\) and \(c_\#\) such that \(f_* = c_*\) and \(f_\# = c_\#\) satisfy both the differential equations and the constraint equations (the constants \(c_*\) and \(c_\#\) should depend on the volume, which equals \(ab\)). In fact, you could do this for any closed manifold with \(R = 0\).

4) On the same torus, suppose \(a \leq b\). For which scales \(\sqrt{\tau}\) is \((M, g)\) \(a\)-noncollapsed? How does this compare with the estimate one might get from part 3 and our work relating log-Sobolev inequalities to noncollapse?