Introduction to Kahler Geometry

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1. Introduction. This is an introduction to Kahler geometry with some of the calculations done as well.

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3. Definitions. An almost-complex manifold M is a smooth manifold with a complex structure J on the tangent space, i.e. $J^2 = -I$. This allows one to make the tangent space T_pM into a complex vector space. The multiplication by i is given by $iV = JV$. We then have a Hermitian structure if we have a Riemannian metric g such that $g(JX, JY) = g(X, Y)$. We have a complex manifold if the almost-complex structure is inherited from a holomorphic structure.

Now, given a complex structure, we can extend our metric to take vectors in $TM \otimes \mathbb{C}$ by simply extending it to be complex linear. Then we have complex coordinates $\{z^\alpha\}$ so that $z^\alpha = x^\alpha + iy^\alpha$. We have a complex structure J on the tangent space so that $J\frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial y^\alpha}$. We then see that the tangent space is spanned by vectors

$$\frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - iJ\frac{\partial}{\partial x^\alpha}\right)$$

$$\frac{\partial}{\partial \overline{z^\alpha}} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + iJ\frac{\partial}{\partial x^\alpha}\right)$$

which are chosen so that $\frac{\partial}{\partial z^\alpha} z^\alpha = 1$ and likewise. Sometimes we will use the notation $\frac{\partial}{\partial z^{\overline{\alpha}}} = \frac{\partial}{\partial \overline{z^\alpha}}$.

Now the complexified tangent space $TM \otimes \mathbb{C}$ splits as a direct sum of holomorphic and antiholomorphic tangent spaces, each diffeomorphic to TM. For instance, the diffeomorphism for holomorphic vectors is:

$$\begin{array}{ccccc}
TM & \to & TM \otimes \mathbb{C} & \to & T^hM \\
\frac{\partial}{\partial x^a} & \mapsto & \left(\frac{\partial}{\partial z^a} + \frac{\partial}{\partial \overline{z^a}}\right) & \mapsto & \frac{\partial}{\partial z^a} \\
J\frac{\partial}{\partial x^a} & \mapsto & i\left(\frac{\partial}{\partial z^a} - \frac{\partial}{\partial \overline{z^a}}\right) & \mapsto & i\frac{\partial}{\partial z^a}
\end{array}$$

Thus we see that if we extend J to act on $TM \otimes \mathbb{C}$, then the holomorphic vectors form the eigenspace with eigenvalue i and the antiholomorphic vectors form the eigenspace with eigenvalue $-i$.

This allows us to show a number of symmetries. We first observe that

$$\begin{aligned} g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right) &= g\left(J\frac{\partial}{\partial z^\alpha}, J\frac{\partial}{\partial z^\beta}\right) \\ &= g\left(i\frac{\partial}{\partial z^\alpha}, i\frac{\partial}{\partial z^\beta}\right) \\ &= -g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right) \end{aligned}$$

thus we get $g_{\alpha\beta} = 0$. Similarly, we get $g_{\overline{\alpha}\overline{\beta}} = 0$.

Proposition 3.1. *The following are true:*

1. $g_{\alpha\overline{\beta}} = g_{\overline{\beta}\alpha}$
2. $g_{\alpha\overline{\beta}} = \overline{g_{\overline{\alpha}\beta}}$
3. *The metric matrix in coordinates* $\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \overline{z^1}}, \dots, \frac{\partial}{\partial \overline{z^n}}\}$ *is of the form*

$$\begin{bmatrix} 0 & A \\ \overline{A} & 0 \end{bmatrix}$$

where A is Hermitian, i.e. $A = \overline{A^T}$.

4. *The metric is*

$$g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \overline{z^\beta}}\right) = \frac{1}{2}g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right) + \frac{i}{2}\omega\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right)$$

where $\omega(X,Y) = g(JX,Y)$ is a symplectic form and $\frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + iJ\frac{\partial}{\partial x^\alpha}\right)$.

Proof:

$$\begin{aligned} g_{\alpha\overline{\beta}} &= g\left(\frac{\partial}{\partial x^\alpha} + iJ\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} - iJ\frac{\partial}{\partial x^\beta}\right) \\ &= g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right) + g\left(J\frac{\partial}{\partial x^\alpha}, J\frac{\partial}{\partial x^\beta}\right) + ig\left(\frac{\partial}{\partial x^\alpha}, J\frac{\partial}{\partial x^\beta}\right) - ig\left(\frac{\partial}{\partial x^\alpha}, J\frac{\partial}{\partial x^\beta}\right) \\ &= 2g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right) + 2ig\left(J\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right) \end{aligned}$$

where we used J-invariance of the metric. We now easily see 1,2, and 4. Now, 3 follows because we let $A_{ab} = g_{a\bar{b}}$ and thus $A^T_{ab} = A_{ba} = g_{b\bar{a}} = \overline{g}_{\bar{b}a} = \overline{g}_{a\bar{b}} = \overline{A}_{ab}$. $\square$

We say the manifold is Kahler if this complex structure is invariant under parallel translation, i.e. $DJ = 0$ (we use D for the covariant derivative). This is equivalent to a number of conditions. Note that when we use capital letters as indices, we mean that the index could be holomorphic or antiholomorphic but when we use lower case indices we indicate antiholomorphic indices with over-bars and holomorphic indices without (so $A = a$ or $\overline{a}$).

Theorem 3.2. *The following are equivalent:*

1. $DJ = 0$
2. $d\omega = 0$
3. $\Gamma^C_{AB} = 0$ *unless all indices are holomorphic or antiholomorpic (Γ is the Levi-Civita connection).*
4. $g_{a\bar{b}} = I_{ab} + O(|z|^2)$ *where I is the identity.*

Proof: First let's prove $1 \Leftrightarrow 2$. $\omega(X, Y) = g(JX, Y)$, so if we work in coordinates $\{x^i\}$ we find:

$$\begin{aligned} d\omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) &= dg\left(J\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\left(\frac{\partial}{\partial x^k}\right) \\ &\quad + dg\left(J\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right)\left(\frac{\partial}{\partial x^i}\right) \\ &\quad + dg\left(J\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i}\right)\left(\frac{\partial}{\partial x^j}\right) \\ &= g\left(D_k\left(J\frac{\partial}{\partial x^i}\right), \frac{\partial}{\partial x^j}\right) + g\left(J\frac{\partial}{\partial x^i}, D_k\frac{\partial}{\partial x^j}\right) \\ &\quad + g\left(D_i\left(J\frac{\partial}{\partial x^j}\right), \frac{\partial}{\partial x^k}\right) + g\left(J\frac{\partial}{\partial x^j}, D_i\frac{\partial}{\partial x^k}\right) \\ &\quad + g\left(D_j\left(J\frac{\partial}{\partial x^k}\right), \frac{\partial}{\partial x^i}\right) + g\left(J\frac{\partial}{\partial x^k}, D_j\frac{\partial}{\partial x^i}\right) \end{aligned}$$

Now suppose 1. Then we have

$$g\left(D_k\left(J\frac{\partial}{\partial x^i}\right), \frac{\partial}{\partial x^j}\right) = g\left(JD_k\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = -g\left(D_k\frac{\partial}{\partial x^i}, J\frac{\partial}{\partial x^j}\right)$$

so using the above, we find

$$\begin{aligned} d\omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) &= g\left(D_i\frac{\partial}{\partial x^k} - D_k\frac{\partial}{\partial x^i}, J\frac{\partial}{\partial x^j}\right) \\ &\quad + g\left(D_j\frac{\partial}{\partial x^i} - D_i\frac{\partial}{\partial x^j}, J\frac{\partial}{\partial x^k}\right) \\ &\quad + g\left(D_k\frac{\partial}{\partial x^j} - D_j\frac{\partial}{\partial x^k}, J\frac{\partial}{\partial x^i}\right) \\ &= 0 \end{aligned}$$

using the symmetry of the Levi-Civita connection (we can do this argument coordinate free, but then we need to recall how to take the exterior derivative of a form in the coordinate-free setting, which will increase the number of terms with lie brackets). This is $1 \Rightarrow 2$.

Now we show $1 \Rightarrow 2$. Now, since our connection is Riemannian and by what we know about the metric tensor it is easy to see that

$$\Gamma^C_{AB} = \overline{\Gamma^{\overline{C}}_{\overline{AB}}}$$

so we can easily reduce 3 to proving that $\Gamma^{\overline{c}}_{ab} = 0$ and $\Gamma^{c}_{a\overline{b}} = 0$

The first is easy:

$$\begin{aligned} \Gamma^{\overline{c}}_{ab} &= \frac{1}{2} g^{\overline{c}d} \left(\frac{\partial}{\partial z^a} g_{bd} + \frac{\partial}{\partial z^b} g_{ad} - \frac{\partial}{\partial z^d} g_{ab} \right) \\ &= 0 \end{aligned}$$

since all the indices of the metric tensor are holomorphic and thus the components are zero.

For the second, we need to use 1. Firstly,

$$\begin{aligned} J\left(D_a \frac{\partial}{\partial z^{\overline{b}}} \right) &= J\left(\Gamma^c_{a\overline{b}} \frac{\partial}{\partial z^c} + \Gamma^{\overline{c}}_{a\overline{b}} \frac{\partial}{\partial z^{\overline{c}}} \right) \\ &= i\Gamma^c_{a\overline{b}} \frac{\partial}{\partial z^c} - i\Gamma^{\overline{c}}_{a\overline{b}} \frac{\partial}{\partial z^{\overline{c}}} \end{aligned}$$

and also, using 1,

$$\begin{aligned} J\left(D_a \frac{\partial}{\partial z^{\overline{b}}} \right) &= D_a \left(J \frac{\partial}{\partial z^{\overline{b}}} \right) \\ &= -i D_a \frac{\partial}{\partial z^{\overline{b}}} \\ &= -i\Gamma^c_{a\overline{b}} \frac{\partial}{\partial z^c} - i\Gamma^{\overline{c}}_{a\overline{b}} \frac{\partial}{\partial x^{\overline{c}}} \end{aligned}$$

and combining these two formulations we see that $\Gamma^c_{a\overline{b}} = 0$, which proves $1 \Rightarrow 2$.

□

4. Curvature. We now want to understand the Riemannian curvature tensor and the Ricci curvature tensor. Recall the definition of the Riemannian curvature tensor in local coordinates:

$$\begin{aligned} R_{ijkl} &= g\left(D_i D_j \frac{\partial}{\partial x^l} - D_j D_i \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k} \right) \\ &= g_{km} \left(\frac{\partial}{\partial x^i} \Gamma^m_{jl} - \frac{\partial}{\partial x^j} \Gamma^m_{il} + \Gamma^p_{il} \Gamma^m_{jp} - \Gamma^p_{jl} \Gamma^m_{ip} \right) \end{aligned}$$

We now have the following:

Theorem 4.1. *The following are true:*

1. $R_{ABCD} = 0$ *if* A *and* B *are both holomorphic or both antiholomorphic.*
2. $R_{a\bar{b}\bar{c}d} = R_{d\bar{b}\bar{c}a} = R_{a\bar{c}\bar{b}d}$
3. $R(X, Y, Z, W) = R(X, Y, JZ, JW)$

Now recall the definition of the Ricci tensor:

$$R_{ij} = g^{kl} R_{ikjl}$$

Notice that the Ricci tensor is symmetric. For our Kahler manifold, we can write this as:

$$\begin{aligned} R_{a\bar{b}} &= g^{\bar{c}d} R_{a\bar{c}\bar{b}d} \\ &= R_{a\bar{c}\bar{b}c} \end{aligned}$$

It is very important to be careful with the Ricci tensor, since we see that

$$R_{a\bar{b}} = R_{c\bar{c}\bar{b}a} = -R_{\bar{c}c\bar{b}a} = R_{\bar{c}ca\bar{b}} = R_{\bar{b}a}$$

(notice again that the Ricci tensor is symmetric) so the moral of the story is to be consistent: we will try to keep the two outside indices the same (conjugated or not) and the two inside indices the same, then we'll be fine.

We also note that the real Ricci curvature is necessarily invariant under J since

$$\begin{aligned} \mathrm{Rc}(X, Y) &= \sum_j \left(R(X, E_j, Y, E_j) + R(X, JE_j, Y, JE_j) \right) \\ &= \sum_j \left(R(JX, JE_j, JY, JE_j) + R(JX, E_j, JY, E_j) \right) \\ &= \mathrm{Rc}(JX, JY) \end{aligned}$$

Now as far as the scalar curvature, we have a bit of a choice. The standard scalar curvature in Riemannian geometry is:

$$R = g^{ij} R_{ij}$$

so on a Kahler manifold, this becomes

$$R = g^{a\bar{b}} R_{a\bar{b}} + g^{\bar{a}b} R_{\bar{a}b} = 2 g^{a\bar{b}} R_{a\bar{b}}$$

but, of course, we might be tempted to define scalar curvature as simply $g^{a\bar{b}} R_{a\bar{b}}$, which is one half the usual scalar curvature.

Let's see how the curvature in the complex coordinates relates to the curvature in real coordinates. Let $z^a = x^a + i y^a$ so that

$$Z_a = \frac{\partial}{\partial z^a} = \frac{1}{2} \left(\frac{\partial}{\partial x^a} - i \frac{\partial}{\partial y^a} \right) = \frac{1}{2} \left(X_a - i Y_a \right)$$

$$\overline{Z_b} = \frac{\partial}{\partial z^{\overline{a}}} = \frac{1}{2}\left(\frac{\partial}{\partial x^a} + i\frac{\partial}{\partial y^a}\right) = \frac{1}{2}\left(X_a + iY_a\right)$$

so $Y_a = JX_a$ and $X_a = -JY_a$.
We now compute:

$$\begin{aligned}
R_{a\overline{b}\overline{c}d} &= R(Z_a, \overline{Z_b}, \overline{Z_c}, Z_d) \\
&= \frac{1}{16}R(X_a - iY_a, X_b + iY_b, X_c + iY_c, X_d - iY_d) \\
&= \frac{1}{16}\big(R(X_a, X_b, X_c, X_d) - iR(X_a, X_b, X_c, Y_d) + iR(X_a, X_b, Y_c, X_d) + R(X_a, X_b, Y_c, Y_d) \\
&\quad + iR(X_a, Y_b, X_c, X_d) + R(X_a, Y_b, X_c, Y_d) - R(X_a, Y_b, Y_c, X_d) + iR(X_a, Y_b, Y_c, Y_d) \\
&\quad - iR(Y_a, X_b, X_c, X_d) - R(Y_a, X_b, X_c, Y_d) + R(Y_a, X_b, Y_c, X_d) - iR(Y_a, X_b, Y_c, Y_d) \\
&\quad + R(Y_a, Y_b, X_c, X_d) - iR(Y_a, Y_b, X_c, Y_d) + iR(Y_a, Y_b, Y_c, X_d) + R(Y_a, Y_b, Y_c, Y_d)\big) \\
&= \frac{1}{4}\left(R(X_a, X_b, X_c, X_d) + R(X_a, Y_b, X_c, Y_d) + iR(X_a, X_b, Y_c, X_d) + iR(X_a, Y_b, X_c, X_d)\right)
\end{aligned}$$

Now, we also look at the metric and it's inverse:

$$\begin{aligned}
g_{a\overline{b}} &= g(Z_a, \overline{Z_b}) \\
&= \frac{1}{4}g(X_a - iY_a, X_b + iY_b) \\
&= \frac{1}{2}\left(g(X_a, X_b) + ig(X_a, Y_a)\right)
\end{aligned}$$

$$\begin{aligned}
g^{a\overline{b}} &= g(dz^a, dz^{\overline{b}}) \\
&= g(dx^a + idy^a, dx^b - idy^b) \\
&= 2\left(g(dx^a, dx^b) - ig(dx^a, dy^b)\right)
\end{aligned}$$

Note that $g^{a\overline{b}}g_{c\overline{b}} = I^a_b$ and $g^{jk}g_{kl} = I^j_l$.
We now wish to compute the hybrid curvature operator:

$$
\begin{aligned}
R_{a\bar{b}}{}^{c\bar{d}} &= R_{a\bar{b}\bar{e}f} g^{c\bar{e}} g^{f\bar{d}} \\
&= \frac{1}{4}\left(\tilde{R}_{abef} + \tilde{R}_{a\tilde{b}e\tilde{f}} + i\tilde{R}_{ab\tilde{e}f} + i\tilde{R}_{a\tilde{b}ef}\right) 2\left(\tilde{g}^{ce} - i\tilde{g}^{c\tilde{e}}\right) 2\left(\tilde{g}^{fd} - i\tilde{g}^{f\tilde{d}}\right) \\
&= \tilde{R}_{abef}\tilde{g}^{ce}\tilde{g}^{fd} - i\tilde{R}_{abef}\tilde{g}^{ce}\tilde{g}^{f\tilde{d}} - i\tilde{R}_{abef}\tilde{g}^{c\tilde{e}}\tilde{g}^{fd} - \tilde{R}_{abef}\tilde{g}^{c\tilde{e}}\tilde{g}^{f\tilde{d}} \\
&\quad + \tilde{R}_{a\tilde{b}e\tilde{f}}\tilde{g}^{ce}\tilde{g}^{fd} - i\tilde{R}_{a\tilde{b}e\tilde{f}}\tilde{g}^{ce}\tilde{g}^{f\tilde{d}} - i\tilde{R}_{a\tilde{b}e\tilde{f}}\tilde{g}^{c\tilde{e}}\tilde{g}^{fd} - \tilde{R}_{a\tilde{b}e\tilde{f}}\tilde{g}^{c\tilde{e}}\tilde{g}^{f\tilde{d}} \\
&\quad + i\tilde{R}_{ab\tilde{e}f}\tilde{g}^{ce}\tilde{g}^{fd} + \tilde{R}_{ab\tilde{e}f}\tilde{g}^{ce}\tilde{g}^{f\tilde{d}} + \tilde{R}_{ab\tilde{e}f}\tilde{g}^{c\tilde{e}}\tilde{g}^{fd} - i\tilde{R}_{ab\tilde{e}f}\tilde{g}^{c\tilde{e}}\tilde{g}^{f\tilde{d}} \\
&\quad + i\tilde{R}_{a\tilde{b}ef}\tilde{g}^{ce}\tilde{g}^{fd} + \tilde{R}_{a\tilde{b}ef}\tilde{g}^{ce}\tilde{g}^{f\tilde{d}} + \tilde{R}_{a\tilde{b}ef}\tilde{g}^{c\tilde{e}}\tilde{g}^{fd} - i\tilde{R}_{a\tilde{b}ef}\tilde{g}^{c\tilde{e}}\tilde{g}^{f\tilde{d}} \\
&= \tilde{R}_{ab}{}^{cd} - i\tilde{R}_{ab}{}^{c\tilde{d}} + i\tilde{R}_{ab}{}^{\tilde{c}d} + \tilde{R}_{ab}{}^{\tilde{c}\tilde{d}} \\
&\quad + \tilde{R}_{a\tilde{b}}{}^{c\tilde{d}} + i\tilde{R}_{a\tilde{b}}{}^{cd} + i\tilde{R}_{a\tilde{b}}{}^{\tilde{c}\tilde{d}} - \tilde{R}_{a\tilde{b}}{}^{\tilde{c}d} \\
&\quad + i\tilde{R}_{ab}{}^{\tilde{c}d} + \tilde{R}_{ab}{}^{\tilde{c}\tilde{d}} + \tilde{R}_{ab}{}^{cd} - i\tilde{R}_{ab}{}^{c\tilde{d}} \\
&\quad + i\tilde{R}_{a\tilde{b}}{}^{cd} + \tilde{R}_{a\tilde{b}}{}^{c\tilde{d}} - \tilde{R}_{a\tilde{b}}{}^{\tilde{c}d} + i\tilde{R}_{a\tilde{b}}{}^{\tilde{c}\tilde{d}} \\
&= 4\left(\tilde{R}_{ab}{}^{cd} + \tilde{R}_{a\tilde{b}}{}^{c\tilde{d}} + i\tilde{R}_{ab}{}^{\tilde{c}d} + i\tilde{R}_{a\tilde{b}}{}^{cd}\right)
\end{aligned}
$$