ANSWERS Final Exam
Math 250b, Section 2 (Professor J. M. Cushing), 15 May 2008

PART 1

1. (10 points) A bacterial population \( x \) grows exponentially according to the equation \( x' = rx \), where \( r > 0 \) is the per unit rate of growth (per minute). We want to eliminate this population by killing the bacteria at a rate \( h \). However, it turns out we cannot sustain the treatment at a constant rate. Instead the treatment’s potency decreases exponentially over time according to the formula \( h(t) = me^{-at} \). Here \( m > 0 \) is the initial (and maximal) potency and \( a > 0 \) is the decay rate of the treatment’s potency over time.

(a) (2 points) The differential equation for \( x \) is \( x' = rx - me^{-at} \).

This follows immediately from the basic rate balance law \( x' = \text{input rate} - \text{output rate} \).

(b) (5 points) Suppose the bacterial population has size \( x_0 > 0 \) at time \( t_0 = 0 \) when the treatment starts. Find a formula for the solution of this initial value problem.

ANSWER: \( x(t) = \left(x_0 - \frac{m}{a+t}\right)e^{rt} + \frac{m}{a+t}e^{-at} \)

The general solution has the form \( x = x_h + x_p \), where \( x_h = ce^{rt} \) is the general solution of the associated homogeneous equation \( x' = rx \). To calculate a particular solution \( x_p \) we can use the Methods of Undetermined Coefficients and let \( x_p = ke^{-at} \). Then \( x' = -ak e^{-at} \) and \( rx - me^{-at} = (rk - m)e^{-at} \) so that we need to choose \( k \) so that \( -ak = rk - m \). Thus \( k = m/(a+r) \) and \( x_p = \frac{m}{a+r}e^{-at} \). Then \( x(0) = c + \frac{m}{a+r} = x_0 \) so that \( c = x_0 - \frac{m}{a+r} \).

(c) (3 points) The treatment succeeds if the bacterial population level \( x(t) \) equals 0 at some time \( t \). Show that the treatment will fail if the initial treatment rate \( m \) is less than a critical value \( m^* \), but will succeed if \( m > m^* \). Find a formula for \( m^* \) and show that it is proportional to \( x_0 \).

ANSWER: The second term in the solution \( x(t) \) tends to 0 as \( t \to +\infty \). The first term is unbounded and tends to \(+\infty \) as \( t \to +\infty \) if \( m < (a+r)x_0 \). In this case, the treatment fails. If \( m > (a+r)x_0 \), then the first term tends to \(-\infty \) as \( t \to +\infty \), and as a result \( x(t) \) equals zero at some time \( t \). Thus, the critical value for \( m \) is \( m^* = (a+r)x_0 \), which is a multiple of \( x_0 \).

2. (15 points) A tumor’s size \( x \) grows at a decreasing per unit rate according to the equation \( x'/x = \frac{1}{1+t} \). Suppose at time \( t_0 = 0 \) the size of the tumor is \( x_0 > 0 \).

(a) (6 points) Find a formula for the solution of the initial value problem for \( t > 0 \) and use it to show that the tumor size grows without bound (and is therefore presumably lethal).

ANSWER: \( x(t) = x_0(t+1) \)

The equation \( x' = \frac{1}{1+t}x \) is linear homogeneous with a general solution \( x(t) = ce^{P(t)} \) where \( P(t) = \int \frac{1}{1+t}dt = \ln(1+t) \). Simplifying, we have \( x(t) = ce^{\ln(1+t)} = c(1+t) \) which equals \( c(t+1) \) for \( t > 0 \). The initial condition implies \( c = x_0 \).
(b) (6 points) Chemotherapy is applied with the result that tumor cells are removed at a constant rate \( r > 0 \), so that \( x' = \frac{1}{t+1} x - r \). Find a formula for the solution of this initial value problem.

**ANSWER:** \( x(t) = (t + 1) (x_0 - r \ln (t + 1)) \)

The solution is given by the Variation of Constants Formula

\[
x(t) = x_0 e^{P(t)} + e^{P(t)} \int_0^t e^{-P(s)} q(s) \, ds
\]

where \( P(t) = \int \frac{1}{t+1} \, dt = \ln (t + 1) \) and hence \( e^{P(t)} = t + 1 \):

\[
x(t) = x_0 (t + 1) + (t + 1) \int_0^t \frac{1}{s+1} (-r) \, ds = x_0 (t + 1) - (t + 1) r \ln (t + 1)
\]

(c) (3 points) Does your answer in (b) show that the chemotherapy treatment will cause the tumor to disappear or not? Explain.

**ANSWER:** Yes. The equation predicts the tumor size reduces to 0 at time \( t = e^{x_0/r} - 1 \)

We ask: does there exist a time \( t > 0 \) such that \( x(t) = (t + 1) (x_0 - r \ln (t + 1)) = 0 \)? In other words, can this equation be solved for \( t > 0 \)? Yes, as the following shows:

\[
x_0 - r \ln (t + 1) = 0 \implies \ln (t + 1) = x_0/r \implies t + 1 = e^{x_0/r} \implies t = e^{x_0/r} - 1.
\]

3. (15 points) Consider the equation: \( x' = (1 - x^2) (1 - e^{1-x}) \).

(a) (4 points) Find all equilibria.

**ANSWER:** \( x_e = -1 \) and 1

\[
(1 - x^2) (1 - e^{1-x}) = 0 \implies (1 - x^2) = 0 \text{ or } (1 - e^{1-x}) = 0
\]

In the first case, \( x = \pm 1 \) and in the second case \( x = 1 \).

(b) (4 points) Which equilibria are hyperbolic? Which are non-hyperbolic?

**ANSWER:** hyperbolic: \( x_e = -1 \) non-hyperbolic: \( x_e = 1 \)

\[
f(x) = (1 - x^2) (1 - e^{1-x}) \implies \left. \frac{df}{dx} \right|_{x_e} = \begin{cases} 2 (1 - e^2) \approx -12.7 & \text{if } x_e = -1 \\ 0 & \text{if } x_e = 1 \end{cases}
\]

(c) (4 points) Classify each equilibrium (sink, source or shunt).

| \( x_e \) | Classification | Since \( \left. \frac{df}{dx} \right|_{x_e} = 2 (1 - e^2) < 0 \) for \( x_e = -1 \), this equilibrium is a sink (by the derivative test or the linearization principle). \( x_e = 1 \) is a shunt because \( f(x) \) does not change sign as \( x \) passes through 1. Or, because \( \frac{d^2 f(x)}{dx^2} = e^{1-x} - 4xe^{1-x} + x^2 e^{1-x} - 2 
\]
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>sink</td>
<td>equals -4 and hence ( f(x) ) has a local maximum at ( x = 1 ).</td>
</tr>
<tr>
<td>1</td>
<td>shunt</td>
<td></td>
</tr>
</tbody>
</table>
(d) (3 points) Draw the phase line portrait.

\[ \rightarrow -1 \leftarrow 1 \leftarrow x \]

4. (10 points) Draw a bifurcation diagram for the equation \( x' = p + (x - 1)^2 (x + 1)^2 \). Locate and identify the type of all bifurcations that occur.

Plot the quartic polynomial \( p = -(x - 1)^2 (x + 1)^2 \) with roots at \pm 1 and then rotate and reflect the graph. The result appears below. There are two saddle-node bifurcations at \( p = 0 \) and another saddle-node bifurcation at \( p = -1 \).

\[ x \]

\[ p \]

\[ \text{PART 2} \]

1. (25 points) The following two-dimensional system of differential equations is a model for a predator-prey system, where \( x \) is the density of the prey and \( y \) the density of the predator. The parameter \( \alpha > 0 \) represents addition of predators at a constant rate.

\[ \frac{dx}{dt} = x(1 - y), \quad \frac{dy}{dt} = y(-1 + x) + \alpha. \] \hfill (1)

(a) (4 points) The point \( P_1 : (x,y) = (1 - \alpha, 1) \) is an equilibrium (fixed point) of the above system. Find any other equilibria.

ANSWER: \( (x,y) = (0, \alpha) \)

The equilibrium equations are

\[ x(1 - y) = 0 \]
\[ y(-1 + x) + \alpha = 0. \]

The first equation implies \( x = 0 \) or \( y = 1 \). In the first case \( x = 0 \), the second equation implies \( y(-1 + 0) + \alpha = 0 \) or \( y = \alpha \). Therefore \( (x,y) = (0,\alpha) \) is an equilibrium. In the second case \( y = 1 \), the second equation implies \( 1 \cdot (-1 + x) + \alpha = 0 \) or \( x = 1 - \alpha \), which gives the equilibrium \( P_1 \).

(b) (8 points) Let \( \alpha = 1/4 \) and classify the equilibrium \( P_1 \) (e.g. saddle, ...). Show all your work.

ANSWER: \( P_1 : (x,y) = (3/4, 1) \) is a stable spiral

The Jacobian

\[ J(x,y) = \begin{pmatrix} 1 - y & -x \\ y & -1 + x \end{pmatrix} \]
evaluated at the equilibrium \((x, y) = (3/4, 1)\)

\[
J(3/4, 1) = \begin{pmatrix} 0 & -3/4 \\ 1 & -1/4 \end{pmatrix}
\]

has characteristic polynomial is \(\lambda^2 + \frac{1}{4}\lambda + \frac{3}{4}\). Since the roots \(\lambda = -\frac{1}{2} \pm \frac{i}{2}\sqrt{47/47} \approx -0.125 + 0.857i\) of this polynomial are complex and have negative real part \(-\frac{1}{2}\), the phase portrait of the linearization is a stable spiral. The Linearization Principle implies that the equilibrium \(P_1\) is also a stable spiral.

(c) (8 points) Show that when \(\alpha = 0\) (i.e. if no predators are added to the system), the linearization at the fixed point \(P_1\), whose coordinates are now \((x = 1, y = 1)\), has a center. In this case (do not try to show this), system (1) has closed orbits (or trajectories) around \(P_1\).

The Jacobian evaluated at the equilibrium \((x, y) = (1, 1)\)

\[
J(1, 1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

has characteristic polynomial is \(\lambda^2 + 1\). Since the roots \(\lambda = \pm i\) of this polynomial are complex with zero real part, the phase portrait of the linearization is a center.

(d) (5 points) Based on (b) above, which of the statements below best describes what happens when predators are added at the constant rate \(\alpha = 1/4\)? Circle your answer. There is no need to explain your reasoning.

**ANSWER:** Both species survive.

Since the phase portrait near the equilibrium \((x, y) = (3/4, 1)\) is a stable spiral, nearby orbits \((x(t), y(t))\), that is to say, as \(t \to \infty\) both species equilibrate at positive (and hence nonzero) values.

2. (5 points) Find a fundamental solution matrix \(\Phi(t)\) for the system \(\dot{x} = Ax\) where

\[
A = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}
\]

and use it to write down the general solution.

**ANSWER:** \(\Phi(t) = \begin{pmatrix} -e^{-2t} & 2e^{3t} \\ e^{-2t} & 3e^{3t} \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -c_1e^{-2t} + 2c_2e^{3t} \\ c_1e^{-2t} + 3c_2e^{3t} \end{pmatrix}\)

The characteristic polynomial: \(\lambda^2 - \lambda - 6\) has roots \(\lambda = -2\) and \(3\). Associated eigenvectors are

\[\lambda = -2 \Rightarrow \hat{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } \lambda = 3 \Rightarrow \hat{w} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}\]
3. (10 points) Calculate the eigenvalues and eigenvectors of the matrices below and use them to identify the type of phase plane portrait associated with the linear system $\dot{x} = Ax$. Also sketch the phase portrait (being sure to include orientation arrows and several typical orbits, including any orbits of any relevant eigen-solutions. (Show your work.)

(a) (4 points) \( A = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \)

\[ \text{TYPE: stable node} \]

The roots of the characteristic polynomial \( \lambda^2 + 4\lambda + 3 \) of \( A \) are \( \lambda = -1, -3 \). Associated eigenvectors are respectively \( \hat{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \) and \( \hat{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Both eigenvalues \( \lambda \) are negative, which implies the portrait is an stable node. Orbits approach the origin tangentially to the direction of \( \hat{v} \), since this vector is associated with the eigenvalue of least magnitude.

(b) (4 points) \( A = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} \)

\[ \text{TYPE: Saddle} \]

The roots of the characteristic polynomial \( \lambda^2 + \lambda - 2 \) of the coefficient matrix are \( \lambda = 1 \) and \( -2 \). Corresponding eigenvectors are, respectively, \( \hat{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \) and \( \hat{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \). The eigenvalues are of opposite sign so the phase portrait is a saddle.
(c) (2 points) Consider the orbit associated with the solution \((x(t), y(t))\) of the system in (b) with initial condition \((x_0, y_0) = (4, -3)\). Sketch the graph of \(x(t)\) and the graph of \(y(t)\). (Plot both on the same axes below.)

The orbit is the hyperbolic shaped orbit in part (b) running from the 4th to the 1st quadrants. Along that orbit \(x\) decreases to nearly 1 after which it increases to approximately 3. \(y\) strictly increases, from -3 to a little more than 4.

![Graph of x(t) and y(t)](image)

4. (10 points) The differential equation for the frictionless mass-spring system studied in class is \(m\ddot{x} + kx = 0\) where \(m > 0\) denotes the mass and \(k > 0\) denotes the spring constant. By dividing by \(m\) we can re-write this equation as \(\ddot{x} + \omega_0^2 x = 0\) where \(\omega_0 = \sqrt{k/m}\). The general solution of this equation \(x(t) = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t\) is periodic with frequency \(\omega_0\), which is called the “natural frequency” of the mass-spring system. If the system is forced with a sinusoidal motion of frequency \(\omega\), the differential equation becomes \(\dddot{x} + \omega_0^2 x = \omega t\). Resonance is said to occur in this system if there are unbounded solutions.

(a) (8 points) Find a formula for the general solution of this equation when the natural frequency \(\omega_0 = 1\). (Use the Method of Undetermined Coefficients twice, once when \(\omega \neq 1\) and once when \(\omega = 1\).)

**ANSWER:** \(x(t) = \begin{cases} c_1 \sin t + c_2 \cos t + (1 - \omega^2)^{-1} \sin \omega t & \text{if } \omega \neq 1 \\ c_1 \sin t + c_2 \cos t - \frac{1}{2} t \cos \omega t & \text{if } \omega = 1 \end{cases} \)

The general solution has the form \(x(t) = x_h(t) + x_p(t)\) where \(x_h(t) = c_1 \sin t + c_2 \cos t\). We use the Method of Undetermined coefficients to find a particular solution of the nonhomogeneous equation of the form \(x_p(t) = k_1 \sin \omega t + k_2 \cos \omega t\) provided \(\omega \neq 1\) (because in that case \(\sin \omega t\) and \(\cos \omega t\) are solutions of the homogeneous equation). If \(\omega = 1\) we take, according to the rules of the Method of Undetermined Coefficients, \(x_p(t) = k_1 \sin t + k_2 \cos t\).

In the first case \(\omega \neq 1\) a calculation shows

\[x_p'' + x_p = (1 - \omega^2) k_1 \sin \omega t + (1 - \omega^2) k_2 \cos \omega t.\]

Therefore, in order to solve the nonhomogeneous equation we must take

\[(1 - \omega^2) k_1 = 1 \text{ and } (1 - \omega^2) k_2 = 0.\]

Because \(\omega^2 \neq 1\) these equations tells us that \(k_1 = (1 - \omega^2)^{-1}\) and \(k_2 = 0\).

In the second case \(\omega = 1\) a calculation shows \(x_p'' + k^2 x_p = 2k_1 \cos t - 2k_2 \sin t.\) Therefore, in order to solve the nonhomogeneous equation we must take \(k_1 = 0\) and \(k_2 = -1/2.\)
(b) (2 points) When the natural frequency of the mass-spring system $\omega_0 = 1$, determine for which values of the external forcing frequency $\omega$ resonance will occur and for which values of $\omega_0$ resonance does not occur. What does this mean physically about the mass-spring system? (Explain your answers.)

ANSWER: Resonance occurs if $\omega = 1$ and does not occur if $\omega \neq 1$.

The general solution in (a) when $\omega \neq 1$ is a sum of bounded functions and is therefore bounded, no matter what the values of the arbitrary constants $c_1, c_2$ are. On the other hand, if $\omega = 1$ the general solution in (a) has an unbounded term (namely, $-\frac{1}{2}t \cos kt$) and is therefore unbounded (no matter what the values of the arbitrary constants $c_1, c_2$ are). Thus, if a mass-spring system is sinusoidally forced with an external frequency equal to the natural frequency of the system, unbounded oscillations will occur that eventually lead to a break down of the system. Other external frequency do not lead to a collapse of the system.