1. [16 points] Let $A$ and $B$ be $3 \times 3$ matrices, with $\det A = 3$ and $\det B = -4$. Find the following:

(a) $\det[A^{-1}BA]$.
(b) $\det[3B^T A]$.
(c) $\det[A^2 B]$.
(d) $\det[(BA^{-1})^{-1}]$
2. [15 points] Solve THREE from the following four questions. Explain your answers!

(a) Determine if the set $H$ of all polynomials of the form $p(t) = 2 + t^2$ is a subspace of $\mathbb{P}_5$.
(b) Determine if the set $H$ of all polynomials in $\mathbb{P}_n$ such that $p(0) = 0$, is a subspace of $\mathbb{P}_n$.

(c) Let $H$ be the set of all matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a$ and $b$ are real numbers. Determine if $H$ is a subspace of $M_{2 \times 2}$.

(d) Let

$$H = \left\{ \begin{bmatrix} a + b \\ b \\ 2a + b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

Determine if $H$ is a subspace of $\mathbb{R}^3$. 

3. [15 points] Let $M_{2\times 2}$ be the vector space of all $2 \times 2$ matrices and define $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A - A^T$, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$ 

(a) Show that $T$ is a linear transformation.
(b) Describe the kernel of $T$.
(c) Find a basis $\mathcal{B}$ of $M_{2\times 2}$, then find $[A]_{\mathcal{B}}$. 


4. [15 points] Find bases for Null $A$ and Col $A$, where

$$A = \begin{bmatrix} 2 & -4 & 2 & 4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$$
5. [15 points] Use coordinate vectors to test the linear independence of the set of polynomials. Explain your work!

\[ 1 + 2t^3, \ 2 + t - 3t^2, \ -t + 2t^2 - t^3. \]
6. [16 points] With no explicit calculations, find one eigenvalue for each of the following matrices, explain your reasoning

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 5 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & 2 \\ 0 & 5 & 1 \\ 0 & -1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & -1 \\ 3 & -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & -1 & 5 \\ 0 & -3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \]
7. [16 points] In the following, \( A \) is an \( n \times n \) matrix. Mark each statement as **True** or **False**.

- ( ) If \( A \) is invertible, then the equation \( A \mathbf{x} = \mathbf{b} \) is consistent for each \( \mathbf{b} \) in \( \mathbb{R}^n \).
- ( ) If \( A \) is invertible, then elementary row operations that reduce \( A \) to the identity \( I_n \) also reduce \( A^{-1} \) to \( I_n \).
- ( ) If \( A^T \) is not invertible then \( A \) is not invertible.
- ( ) If the columns of \( A \) span \( \mathbb{R}^n \) then they are linearly independent.
- ( ) If the columns of \( A \) are linearly independent then they span \( \mathbb{R}^n \).
- ( ) If \( A \) is invertible then the columns of \( A^{-1} \) are linearly independent.
- ( ) If \( A \) is invertible then the inverse of \( 2A^TA^{-1} \) is \( 2A(A^{-1})^T \).
- ( ) If \( \det A \) is zero, then two rows or two columns are the same, or a row or a column is zero.
- ( ) \( \det(rA) = r \det A \) where \( r \) is in \( \mathbb{R} \).
- ( ) The range of a linear transformation is a vector space.
- ( ) Let \( B \) be an \( n \times m \) matrix. The set of all solutions of the homogeneous system \( B\mathbf{x} = \mathbf{0} \) is a subspace of \( \mathbb{R}^m \).
- ( ) If \( A \) is invertible then the number 0 is not an eigenvalue of \( A^T \).
- ( ) If \( H = \text{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\} \) then \( \{\mathbf{b}_1, \ldots, \mathbf{b}_p\} \) is a basis for \( H \).
- ( ) A basis is a spanning set that is as large as possible.
- ( ) If a finite set \( S \) of nonzero vectors spans a vector space \( V \), then some subset of \( S \) is a basis for \( V \).
- ( ) The number of free variable in the equation \( A\mathbf{x} = \mathbf{0} \) equals the dimension of \( \text{Null} \ A \).