1. [15 points] Suppose that $U$ is a 3-dimensional subspace of $\mathbb{R}^8$ and that $T \in \mathcal{L}(\mathbb{R}^8, \mathbb{R}^5)$ such that $\text{null}(T) = U$. Prove that $T$ is surjective.
2. [15 points] $V$ is a finite dimensional vector space over a field $\mathbb{F}$. Let $T \in \mathcal{L}(V)$, prove that if $U_1$ and $U_2$ are two invariant subspaces under $T$ then

(a) $U_1 + U_2$ is invariant under $T$.
(b) $U_1 \cap U_2$ is invariant under $T$. 
3. [15 points] Let $V$ and $W$ be vector spaces over $F$, and suppose $T \in \mathcal{L}(V,W)$ is surjective. Given that $V = \text{span}(v_1, \ldots, v_n)$, prove that $W = \text{span}(Tv_1, \ldots, Tv_n)$. 
4. [15 points] Let $V$ be a finite dimensional vector space over a field $F$, and let $S, T \in \mathcal{L}(V)$. Prove that $T \circ S$ is invertible if and only if both $S$ and $T$ are invertible.
5. [15 points] Let $V$ be a finite dimensional vector space over a field $F$ and let $T_j \in \mathcal{L}(V)$ for $j = 1, 2, \ldots, k$. Suppose that there is a non-zero vector $v \in V$ and a constant $\lambda \in F$ such that

$$(T_1 - \lambda \mathbb{1})(T_2 - \lambda \mathbb{1}) \ldots (T_k - \lambda \mathbb{1}) v = 0$$

Prove that there exists a $j \in \{1, 2, \ldots, k\}$ such that $\lambda$ is an eigenvalue of $T_j$. 
6. [15 points] Let $V$ be a finite dimensional vector space over the field $\mathbb{F}$, and let $\phi \in \mathcal{L}(V, \mathbb{F})$. Suppose $u \in V$ is not in $\text{null}(\phi)$. Prove that

\[ V = \text{null}(\phi) \oplus \text{span}\{u\}. \]
7. [15 points] Let $V$ be a finite dimensional vector space over the field $\mathbb{F}$, and let $T \in \mathcal{L}(V)$ such that every $v \in V$ is an eigenvector for $T$. Prove that there exists an $\alpha \in \mathbb{F}$ such that $T = \alpha \mathbb{I}$. 