(1) Let $F_m[z]$ denote the vector space of all polynomials with degree less than or equal to $n \in \mathbb{Z}_+$ and having coefficient over $F$, and suppose that $p_0, p_1, \ldots, p_m \in F_m[z]$ satisfy $p_j(1) = 0$. Prove that $(p_0, p_1, \ldots, p_m)$ is a linearly dependent list of vectors in $F_m[z]$.

(2) Let $V$ be a finite dimensional vector space over the field $F$, and suppose that $P \in \mathcal{L}(V)$ has the property that $P^2 = P$. Prove that $V = \text{null}(P) \oplus \text{range}(P)$.

(3) Solve for $x$ the following equation

$$\det\left(\begin{bmatrix} x & -1 \\ 3 & 1 - x \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x - 5 \end{bmatrix}\right)$$

(4) Let $A$ be a square matrix. Prove that $A$ is invertible if and only if $A^T A$ is invertible.

(5) Suppose that $V$ is a real inner product space.

(a) Show that if $u, v \in V$ have the same norm, then $u + v$ is orthogonal to $u - v$.

(b) Use part (a) to show that the diagonals of a rhombus are perpendicular to each other.

(6) Suppose that $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for all $v \in V$. Prove that $(T - 2I)$ is invertible.

(7) Suppose $(V, \langle \cdot, \cdot \rangle)$ is an inner product space. Let $T \in \mathcal{L}(V)$ is an injective operator on $V$. Define $\langle \cdot, \cdot \rangle_1$ by

$$\langle u, v \rangle_1 := \langle Tu, Tv \rangle,$$

for all $u, v \in V$. Show that $\langle \cdot, \cdot \rangle_1$ is an inner product on $V$. 

Due on: Friday 06-22-2018.