1 Let $V$ be a finite dimensional vector space over $\mathbb{C}$ with $T \in \mathcal{L}(V)$. Prove that, for each $k = 1, 2, \ldots, \dim(V)$, there is a $T$-invariant subspace $U_k$ of $V$ such that $\dim(U_k) = k$.

2 Let $A$ be an invertible matrix in $\mathbb{R}^{n \times n}$, suppose that $A^2 - 2A = 0$, find $\det(2A^T A^2)$.

3 Let $(e_1, e_2, e_3)$ be the canonical basis of $\mathbb{R}^3$, and define $f_1 = e_1 + e_2 + e_3, \ f_2 = e_2 + e_3, \ f_3 = e_3$

(a) Apply the Gram-Schmidt process to the basis $(f_1, f_2, f_3)$.
(b) What do you obtain if you instead applied the Gram-Schmidt process to the basis $(f_3, f_2, f_1)$?

4 Let $\mathbb{R}_2[x]$ be the inner product space of polynomials over $\mathbb{R}$ having degree at most two, with inner product given by $\langle f, g \rangle = \int_0^1 f(x)g(x) \ dx$, for every $f, g \in \mathbb{R}_2[x]$.

Apply the Gram-Schmidt procedure to the standard basis $\{1, x, x^2\}$ for $\mathbb{R}_2[x]$ in order to produce an orthonormal basis for $\mathbb{R}_2[x]$.

5 Let $n \in \mathbb{Z}_+$, and let $a_1, a_2, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ be any collection of $2n$ real numbers. Prove that

$$\left( \sum_{k=1}^n a_k b_k \right)^2 \leq \left( \sum_{k=1}^n k a_k^2 \right) \left( \sum_{k=1}^n b_k^2 \right).$$

Hint: Use Cauchy-Schwarz inequality with the “correct” choice of vectors.

6 Let $V$ be a finite dimensional inner product space over $\mathbb{R}$. Given $u, v \in V$, prove that $\langle u, v \rangle = \frac{1}{4} \left( \|u + v\|^2 - \|u - v\|^2 \right)$.

7 Let $V$ be a finite dimensional vector inner product space over $\mathbb{F}$, and suppose that $P \in \mathcal{L}(V)$ with $P^2 = P$ and $\text{null}(P) = (\text{range}(P))^\perp$. Prove that $P$ is an orthogonal projection.