(1) Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}[x] \to \mathbb{R}^2$ by
\[
Tp = \left(3p(4) + 5p'(6) + bp(2), \int_{-1}^{2} x^3 p(x) \, dx + c \sin(p(0))\right).
\]
Show that $T$ is linear if and only if $b = c = 0$.

(2) Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in $V$ such that $(Tv_1, \ldots, Tv_m)$ is linearly independent list of vectors in $W$. Prove that $v_1, v_2, \ldots, v_m$ are linearly independent.

(3) Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that
\[
f(\alpha v) = \alpha f(v)
\]
for all $\alpha \in \mathbb{R}$ and $v \in \mathbb{R}^2$, but $f$ is not linear.

(4) Check that the function $f : \mathbb{C} \to \mathbb{C}$ defined as
\[
f(z) = \Re z
\]
satisfies
\[
f(u + v) = f(u) + f(v)
\]
for all $u, v \in \mathbb{C}$, but $f$ is not linear.

(5) Suppose $V$ is a vector space and $S, T \in \mathcal{L}(V)$ are such that
\[
\text{range } S \subseteq \text{null } T.
\]
Prove that $(ST)^2 = 0$.

(6) Show that
\[
U := \{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4); \dim(\text{null } T) > 2\}
\]
is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$. \textit{Hint: Give an example of two linear maps $S$ and $T$ in $U$ such that $S + T$ is not in $U$.}

(7) Give an example of a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that
\[
\text{range } T = \text{null } T.
\]

(8) Prove that there does not exist a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that
\[
\text{range } T = \text{null } T.
\]

(9) Suppose that $V$ is finite dimensional vector space and let $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace $U$ of $V$ such that $U \cap \text{null } T = \{0\}$ and $\text{range } T = \{Tu; \ u \in U\}$. 