In the following, $U$, $V$, and $W$ are finite dimensional vector spaces over a field $F$.

1. Suppose $U$ is 3-dimensional subspace of $\mathbb{R}^8$ and $T$ is a linear map from $\mathbb{R}^8$ to $\mathbb{R}^5$ such that $\text{null } T = U$. Prove that $T$ is surjective.

2. Suppose that $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$ are both injective. Prove that $T \circ S$ is injective.

3. Suppose $S, T \in \mathcal{L}(V)$. Prove that $T \circ S$ is invertible if and only if both $S$ and $T$ are invertible.

4. Let $S, T \in \mathcal{L}(V)$. Prove that $S \circ T = \mathbb{I}$ if and only if $T \circ S = \mathbb{I}$.

5. Let $R, S, T \in \mathcal{L}(V)$. Prove that if $RST$ is surjective then $S$ is injective.

6. For any positive integers $m, n$, $\mathbb{F}^{m,n}$ is used to denote the set of all $m \times n$ matrices with entries from the field $\mathbb{F}$. It is direct to check that $\mathbb{F}^{m,n}$ is a vector space under the obvious matrix addition and scalar multiplication. Find a basis for $\mathbb{F}^{2,2}$. What is the dimension of $\mathbb{F}^{m,n}$?

7. Suppose $v_1, \ldots, v_n$ is a basis for $V$. Prove that the map $T : V \to \mathbb{F}^{n,1}$ defined by
   \[ T v = \mathcal{M}(v) \]
   is an isomorphism of $V$ onto $\mathbb{F}^{n,1}$ where $\mathcal{M}(v)$ is the matrix of $v \in V$ with respect to the basis $v_1, \ldots, v_m$.

8. Suppose $\phi \in \mathcal{L}(V, \mathbb{F})$. Suppose $u \in V$ is not in $\text{null}(\phi)$. Prove that
   \[ V = \text{null}(\phi) \oplus \text{span}\{u\} \]

9. Let $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$. Prove that
   \[ \dim(\text{null}(T \circ S)) \leq \dim(\text{null}(T)) + \dim(\text{null}(S)) \]